Plane Algebraic Curves Drawn by the Orthocenter of a Pedal Triangle

— Applications of a Drawing Tool and Mathematica —

by

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Abstract

In this paper we present a computer tool for drawing a locus of the orthocenter of the pedal triangle for a triangle with two vertices fixed when one moves the third vertex along a distinguished curve. The drawing tool provides many plane algebraic curves with simple expressions.

§ 1. Introduction

In the sequel to [4]-[7], [14] our study aims to develop a drawing tool on a display for experimental research on various curves using computers.

In elementary geometry we have five significant notions for a triangle; that is, the center of gravity, the center of an inscribed circle, the center of an escribed circle, the circumcenter and the orthocenter of a triangle.

Which curve is drawn as a locus of such a point of a pedal triangle for a triangle with two vertices fixed on the plane when the third vertex moves under a certain condition? In this study we limit ourselves to the case where two vertices of a triangle
are fixed while the third vertex moves along a distinguished curve. Then our main concern is to find various unknown curves with simple algebraic expressions as a locus of the orthocenter of a pedal triangle.

For terminology of geometry throughout the paper, consult [2]-[3] and [16].

§ 2. A program for drawing a locus

Let us consider a triangle $\triangle ABC$ given on a plane.

Let $A = (x_a, y_a), B = (x_b, y_b), C = (x_c, y_c)$. Then the coordinates of the orthocenter $H$ of $\triangle ABC$ is given by

\[(*) \quad \left( \frac{r_1q_2 - r_2q_1}{p_1q_2 - p_2q_1}, \frac{p_1r_2 - p_2r_1}{p_1q_2 - p_2q_1} \right),\]

where

\[
p_1 = x_b - x_c, \quad q_1 = y_b - y_c, \quad r_1 = x_a(x_b - x_c) + y_a(y_b - y_c),
\]

\[
p_2 = x_a - x_c, \quad q_2 = y_a - y_c, \quad r_2 = x_b(x_a - x_c) + y_b(y_a - y_c).
\]

Let $D, E, F$ be the feet of the perpendiculars from the vertices $A, B, C$ to the opposite sides, respectively. The $\triangle DEF$ is called a pedal triangle for $\triangle ABC$. Denote the orthocenter of $\triangle DEF$ by $K$.

Which curve is drawn as a locus of the orthocenter $K$ of the pedal triangle $\triangle DEF$ for $\triangle ABC$ with the vertices $B, C$ fixed when the third vertex $A$ moves along a distinguished curve $C$?

We divide our operation of a drawing tool into the drawing part and the printing part, due to the circumstances of our computer machines.

**PART ONE**: To draw a locus of the orthocenter of a pedal triangle for a given triangle. A program of our drawing game for Case 11 of Theorem in § 3 is written in Visual Basic Ver. 6.0 by Microsoft Corporation and consists of the following seven steps (see List 1).

**Step 1.** Set the coordinates axes and two fixed point $B, C$ in black and a curve $C$ in blue.

**Step 2.** Set the initial position of a vertex $A$ of a triangle $\triangle ABC$ and draw each side
in black.

**Step 3.** Plot each foot $D$, $E$, $F$ of the perpendiculars from the vertices of $\triangle ABC$ to the opposite sides in green.

**Step 4.** Draw the sides of the pedal triangle $\triangle DEF$ for $\triangle ABC$ in green and plot its orthocenter $K$ in red.

**Step 5.** Draw each of the perpendiculars from the vertices of $\triangle DEF$ to the opposite sides with a broken line in blue.

**Step 6.** When $\triangle DEF$ is obtuse, extend each of two sides adjacent at the obtuse angle with a broken line in blue.

**Step 7.** When one moves the vertex $A$ continuously along the curve $C$ by the mouse, the orthocenter $K$ of $\triangle DEF$ continuously draws a locus $\mathcal{L}$ with a solid line in red.

**Part Two:** To process a bitmap file (bmp) by \LaTeX{}, to exhibit it on another display, and to print it.

**Outline of the Program.** Let $B(-1, 0)$, $C(1, 0)$ be the fixed vertices of a triangle on the plane, and let $C$ be the straight line $y = x$. When a point $A(u, v)$ on $C$ is selected by the mouse, then the orthocenter $K(x, y)$ of the $\triangle DEF$ is determined by the formula (*) with $D = (x_a, y_a)$, $E = (x_b, y_b)$, $F = (x_c, y_c)$.

The program for drawing a locus $\mathcal{L}$ of the point $K$ is given in List 1. In this case we will have the curve

$$\mathcal{L} : 4xy^4 + 8x(x^2 + 3)y^2 - 4(6x^2 - 1)y + x(2x^2 + 2x - 1)(2x^2 - 2x - 1) = 0$$

on a display (Fig. 11).

We keep the same notations as in this section throughout the paper.
§ 3. Algebraic Curves obtained as locus of the orthocenter of a pedal triangle

As a locus of the orthocenter of a pedal triangle for a given triangle we have various curves. In this section we exhibit the cases of plane algebraic curves with simple expressions whose graphs are not given in the standard texts of the subject [8]-[9], [11]-[13], [15] and [17]-[18].

Let \( B(-1, 0), \ C(1, 0) \) be the vertices fixed throughout this section.

**Theorem.** Each of the following algebraic curves \( \mathcal{C} \) can be obtained as a locus of the orthocenter of the pedal triangle \( \triangle DEF \) for \( \triangle ABC \) when the third vertex \( A \) moves along the conic section \( \mathcal{C} \).

**Case 1-1 (Fig. 1-1)**
\[
\mathcal{C} : \ y = 1, \ \text{the straight line}.
\]
\[
\mathcal{L} : \ x^2y^4 + 2x^2y^3 + 2x^2(x^2 + 4)y^2 + 2(x^4 - 4x^2 - 4)y + (x^2 + 2)(x^2 - 2)^2 = 0.
\]

**Case 1-2 (Fig. 1-2)**
\[
\mathcal{C} : \ y = -x^2 + 1, \ \text{the parabola}.
\]
\[
\mathcal{L} : \ x^3y^4 + 2x^3y^3 + 2x^2(x^2 + 4)y^2 + 2(x^4 - 4x^2 - 4)y + (x^2 + 2)(x^2 - 2)^2 = 0.
\]

**Case 2 (Fig. 2)**
\[
\mathcal{C} : \ x = y^2 + 1, \ \text{the parabola}.
\]
\[
\mathcal{L} : \ 4x^3y^4 + x^2(5x^3 + 24x^2 + 68x + 88)y^2 + (x - 2)(x^3 + 3x^2 - 8)^2 = 0.
\]

**Case 3 (Fig. 3)**
\[
\mathcal{C} : \ x = y^2 - 1, \ \text{the parabola}.
\]
\[
\mathcal{L} : \ x^3y^6 + x^2(3x^3 + 2x^2 + 9x + 18)y^4
+ (3x^7 + 4x^6 + 7x^5 - 14x^4 - 16x^3 - 16x^2 + 48x - 32)y^2
+ (x + 2)(x - 1)^2(x^3 + x^2 - 4)^2 = 0.
\]

**Case 4 (Fig. 4)**
\[
\mathcal{C} : \ y^2 - x^2 = 7, \ \text{the hyperbola}.
\]
\[
\mathcal{L} : \ 4x^3y^6 + x^2(16x^2 - 15)y^4 + 2(12x^6 + 5x^4 + 143x^2 + 56)y^2
+ (16x^8 + 65x^6 + 2x^4 - 479x^2 - 4)y^2 + 4x^2(x^4 + 5x^2 + 2)^2 = 0.
\]
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Case 5 (Fig. 5)
\[ C : y^2 - x^2 = 2, \text{ the hyperbola.} \]
\[ L : 144x^2y^2 + 64x^2(9x^2 + 10)y^2 + 8(108x^6 + 160x^4 + 257x^2 + 144)y^4 + 16(36x^8 + 40x^6 - 3x^4 - 114x^2 - 49)y^2 + 9x^2(4x^2 - 7)^2 = 0. \]

Case 6 (Fig. 6)
\[ C : x^2 - 3y^2 = 1, \text{ the hyperbola.} \]
\[ L : 16x^2y^2 + 4(8x^4 + 70x^2 - 3)y^2 + (x + 2)(x - 2)(4x^2 - 7)^2 = 0. \]

Case 7 (Fig. 7)
\[ C : (x + 2)^2 - y^2 = 1, \text{ the hyperbola.} \]
\[ L : xy^6 + (3x^2 + 8x^2 + 6x + 6)y^4 + (3x^5 + 16x^4 + 31x^3 + 20x^2 - 15x - 26)y^2 + (x + 2)(x^3 + 3x^2 + 2x - 2)^2 = 0. \]

Case 8 (Fig. 8)
\[ C : x^2 - y^2 = 2, \text{ the hyperbola.} \]
\[ L : 16x^2y^2 + 2x^2(32x^3 + 384)y^2 + 4(24x^6 + 352x^4 + 1762x^2 - 32)y^4 + 16(4x^8 + 32x^6 - 211x^4 + 400x^2 - 289)y^2 + x^2(4x^2 - 16x^2 + 17)^2 = 0. \]

Case 9 (Fig. 9)
\[ C : (2x + 1)^2 - 4y^2 = 1, \text{ the hyperbola.} \]
\[ L : 16xy^6 + 8x(6x^2 + 4x + 15)y^4 + (48x^2 + 64x^2 + 64x^3 - 424x^2 + 81x - 8)y^2 + (x + 2)(4x^3 - 7x + 1)^2 = 0. \]

Case 10-1 (Fig. 10-1)
\[ C : y = 2x + 2, \text{ the straight line.} \]
\[ L : 125xy^2 + 120y + (5x + 6)(5x + 7)(5x - 1) = 0. \]

Case 10-2 (Fig. 10-2)
\[ C : y = \frac{1}{2}x + \frac{1}{2}, \text{ the straight line.} \]
\[ L : 125xy^2 - 120y + (5x - 6)(5x - 7)(5x + 1) = 0. \]
Case 11 (Fig. 11)
\[ C : y = x, \quad \text{the straight line.} \]
\[ L : 4xy^4 + 8x(x^2 + 3)y^3 - 4(6x^2 - 1)y + x(2x^2 + 2x - 1)(2x^2 - 2x - 1) = 0. \]

Case 12 (Fig. 12)
\[ C : x^2 + y^2 = 3, \quad \text{the circle.} \]
\[ L : 4x^2y^6 + x^2(9x^2 + 56)y^4 + (6x^6 - 27x^4 + 52x^2 - 48)y^2 + (x^2 - 2)^4 = 0. \]

Case 13 (Fig. 13)
\[ C : (x - 1)^2 + y^2 = 1, \quad \text{the circle.} \]
\[ L : 64x^2y^6 + 64x(3x^3 + 3x^2 + 16x - 4)y^4 + 16x(12x^5 + 32x^4 - 90x^3 - 30x^2 + 133x - 120)y^2 + (x^2 + 4x - 8)^2(8x^2 - 12x + 1)^2 = 0. \]

Case 14 (Fig. 14)
\[ C : \left( x - \frac{1}{2} \right)^2 + y^2 = \left( \frac{1}{2} \right)^2, \quad \text{the circle.} \]
\[ L : 64xy^4 + x(89x^2 + 400x + 464)y^2 + (x - 2)(x + 4)^2(5x - 1)^2 = 0. \]

Case 15 (Fig. 15)
\[ C : (x - 1)^2 + (y - 2)^2 = 4, \quad \text{the circle.} \]
\[ L : xy^4 + 2x(2x - 1)y^3 + 2(3x^3 + 3x - 1)y^2 + 2(2x^2 + 3x^2 + 6x^2 - x - 2)y + (x + 1)(5x + 1)(x^3 + 2x^2 + x - 2) = 0. \]

Case 16 (Fig. 16)
\[ C : (x - 3)^2 + y^2 = 4, \quad \text{the circle.} \]
\[ L : xy^4 + 2(13x^3 - 25x^2 + 23x - 5)y^2 + (x - 2)(x - 1)^2(5x - 7)^2 = 0. \]

Case 17 (Fig. 17)
\[ C : (x - 1)^2 + y^2 = 2, \quad \text{the circle.} \]
\[ L : x^3y^6 + 4x(3x^3 - 2x^2 + 14x - 4)y^4 + 4(3x^6 - 18x^4 + 24x^3 - 2x^2 - 8x - 4)y^2 + (x^2 + 2x - 4)^2(2x^2 - 2x - 1)^2 = 0. \]

Case 18 (Fig. 18)
\[ C : (x - 2)^2 + y^2 = 9, \quad \text{the circle.} \]
\[ L : 36xy^4 + (261x^3 - 1200x^2 + 1216x - 360)y^2 + (x + 2)(3x - 2)^2(5x - 1)^2 = 0. \]
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Case 19-1 (Fig. 19-1)
\[ C : \quad x^2 + 5y^2 = 1, \quad \text{the ellipse.} \]
\[ L : \quad 144x^2 y^4 + (288x^4 + 2200x^2 + 1620)y^2 + 9(x + 2)(x - 2)(4x^2 + 1)^2 = 0. \]

Case 19-2 (Fig. 19-2)
\[ C : \quad x^2 + \frac{1}{5}y^2 = 1, \quad \text{the ellipse.} \]
\[ L : \quad 144x^2 y^4 + (288x^4 + 2200x^2 + 1620)y^2 + 9(x + 2)(x - 2)(4x^2 + 1)^2 = 0. \]

Case 20-1 (Fig. 20-1)
\[ C : \quad x^2 + 7y^2 = 1, \quad \text{the ellipse.} \]
\[ L : \quad 324x^2 y^4 + (648x^4 + 4185x^2 + 1792)y^2 + 4(x + 2)(x - 2)(9x^2 - 2)^2 = 0. \]

Case 20-2 (Fig. 20-2)
\[ C : \quad x^2 + \frac{1}{7}y^2 = 1, \quad \text{the ellipse.} \]
\[ L : \quad 324x^2 y^4 + (648x^4 + 4185x^2 + 1792)y^2 + 4(x + 2)(x - 2)(9x^2 - 2)^2 = 0. \]

**Proof.** Let \( A(u, v) \) and \( K(x, y) \), while \( B = (-1, 0) = (-\alpha, -\beta) \), \( C = (1, 0) = (\alpha, -\beta) \) are fixed. Then we have the following:

\[ AC : \quad y + \beta = \frac{v + \beta}{u - \alpha}(x - \alpha), \]

and

\[ BE : \quad y + \beta = -\frac{u - \alpha}{v + \beta}(x + \alpha). \]

Hence

\[ E = \left( \frac{-(u - 1)^2 + v^2}{(u - 1)^2 + v^2}, \quad \frac{-2v(u - 1)}{(u - 1)^2 + v^2} \right), \]

\[ F = \left( \frac{(u + 1)^2 - v^2}{(u + 1)^2 + v^2}, \quad \frac{2v(u + 1)}{(u + 1)^2 + v^2} \right). \]

since \( \alpha = 1, \beta = 0. \)
**Case 9.** Find the Groebner bases (see [1], [10]) by applying Mathematica Ver 3.0 by Wolfram Research Inc., to the following code, where \( D = (x_a, y_a), \ E = (x_b, y_b), \ F = (x_c, y_c) \):

\[
\begin{align*}
f0 & := (2u+1)^2-4v^2-1 \\
f1 & := xa-u \\
f2 & := ya \\
f3 & := ((u-1)^2+2v^2)xb-(v^2-(u-1)^2) \\
f4 & := ((u-1)^2+2v^2)yb+2(u-1)v \\
f5 & := ((u+1)^2+2v^2)xc-((u+1)^2-v^2) \\
f6 & := ((u+1)^2+2v^2)yc-2(u+1)v \\
f7 & := p1-(xc-xb) \\
f8 & := q1-(yc-yb) \\
f9 & := r1-((xc-xb)*xa+(yc-yb)*ya) \\
f10 & := p2-(xa-xc) \\
f11 & := q2-(ya-yc) \\
f12 & := r2-((xa-xc)*xb+(ya-yc)*yb) \\
f13 & := (p1*q2-p2*q1)*x-(r1*q2-r2*q1) \\
f14 & := (p1*q2-p2*q1)*y-(p1*r2-p2*r1)
\end{align*}
\]

GroebnerBasis\[\{f0,f1,f2,f3,f4,f5,f6,f7,f8,f9,f10,f11,f12,f13,f14\}\],
\{xa, ya, xb, yb, xc, yc, p1, p2, q1, q2, r1, r2, v, u, y, x\}\]

\[
\text{Factor}[\%]
\]

Then we obtain the term

\[
(-1+u)(1+u)^2(-1+2u)(2-27x+84x^2+65x^3-104x^4-56x^5+32x^6
\]
\[
+16x^7-8y^2+81xy^2-424x^2y^2+64x^3y^2+64x^4y^2+48x^5y^2+120xy^4
\]
\[
+32x^2y^4+48x^3y^4+16xy^6)
\]

in the list of the generating Groebner bases. From the last factor we have

\[
16xy^6+8x(6x^3+4x^2+15)y^4
\]
\[
+(48x^5+64x^4+64x^3-424x^2+81x-8)y^2+(x+2)(4x^3-7x+1)^2=0
\]

as an equation of \(L\).

**Case 11.** Find the Groebner bases by applying Mathematica Ver 3.0 to the following code. At first we put \( f0:=u^2-v^2 \), because the calculation by Mathematica stopped for the setting \( f0:=u-v \) which means the equation of \( C : y=x \).
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\[ f_0 := u^2 - v^2 \]
\[ f_1 := x_a - u \]
\[ f_2 := y \]
\[ f_3 := ((u-1)^2 + v^2) x_b - (v^2 - (u-1)^2) \]
\[ f_4 := ((u-1)^2 + v^2) y_b + 2*(u-1)*v \]
\[ f_5 := ((u+1)^2 + v^2) x_c - ((u+1)^2 - v^2) \]
\[ f_6 := ((u+1)^2 + v^2) y_c - 2*(u+1)*v \]
\[ f_7 := p_1 - (x_c - x_b) \]
\[ f_8 := q_1 - (y_c - y_b) \]
\[ f_9 := r_1 - ((x_c - x_b)*x_a + (y_c - y_b)*y_a) \]
\[ f_{10} := p_2 - (x_a - x_c) \]
\[ f_{11} := q_2 - (y_a - y_c) \]
\[ f_{12} := r_2 - ((x_a - x_c)*x_b + (y_a - y_c)*y_b) \]
\[ f_{13} := (p_1*q_2 - p_2*q_1)*x - (r_1*q_2 - r_2*q_1) \]
\[ f_{14} := (p_1*q_2 - p_2*q_1)*y - (p_1*r_2 - p_2*r_1) \]
\[ \text{GroebnerBasis}\{f_0, f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9, f_{10}, f_{11}, f_{12}, f_{13}, f_{14}\}, \]
\[ \{x_a, y_a, x_b, y_b, x_c, y_c, p_1, p_2, q_1, q_2, r_1, r_2, v, u, y, x\} \]
\[ \text{Factor}[] \]

Then we obtain the term

\[ ( - 1 + u)(1 + u)(- 1 + 2u^2)(x - 8x^3 + 4x^5 + 4y - 24x^2y + 24xy^2 + 8x^3y^2 + 4xy^4) \]
\[ \times (x - 8x^3 + 4x^5 - 4y + 24x^2y + 24xy^2 + 8x^3y^2 + 4xy^4) \]

in the list of the generating Groebner bases. From the last two factors we have

\[ x - 8x^3 + 4x^5 + 4x - 24x^2y + 24xy^2 + 8x^3y^2 + 4xy^4 = 0, \]

or

\[ x - 8x^3 + 4x^5 - 4y + 24x^2y + 24xy^2 + 8x^3y^2 + 4xy^4 = 0. \]

as two candidates for an equation of \( L \). By checking their graphs we conclude that the former gives a required equation; that is,

\[ 4xy^4 + 8x(x^2 + 3)y^2 - 4(6x^2 - 1)y + x(2x^2 + 2x - 1)(2x^2 - 2x - 1) = 0. \]

The proofs for the rest cases are omitted. \( \square \)
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§ 4. Questions

Question 1. The two curves $\mathcal{L}$ in the following cases coincide with each other:

i) Cases 1-1 and 1-2.
ii) Cases 19-1 and 19-2.
iii) Cases 20-1 and 20-2.

Why is it so?

Question 2. The two curves $\mathcal{L}$ in Case 10-1 and 10-2 are reflexive in the $y$-axis. Why is it so?

References


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Fig. 1-1 \( \mathcal{C} \): \( y = 1 \)

\[ \mathcal{L} : x^3y^4 + 2x^2y^3 + 2x^2(x^2 + 4)y^2 + 2(x^4 - 4x^2 - 4)y + (x^2 + 2)(x^2 - 2)^2 = 0 \]

Fig. 1-2 \( \mathcal{C} \): \( y = -x^2 + 1 \)

\[ \mathcal{L} : x^3y^4 + 2x^2y^3 + 2x^2(x^2 + 4)y^2 + 2(x^4 - 4x^2 - 4)y + (x^2 + 2)(x^2 - 2)^2 = 0 \]

Fig. 2 \( \mathcal{C} \): \( x = y^2 + 1 \)

\[ \mathcal{L} : 4x^3y^4 + x^2(5x^3 + 24x^2 + 68x + 88)y^2 + (x - 2)(x^3 + 3x^2 - 8)^2 = 0 \]

Fig. 3 \( \mathcal{C} \): \( x = y^2 - 1 \)

\[ \mathcal{L} : x^3y^8 + x^7(3x^3 + 2x^2 + 9x + 18)y^4 + (3x^7 + 4x^6 + 7x^5 - 14x^4 + 16x^3 - 16x^2 + 48x - 32)y^2 + (x + 2)(x - 1)^2(x^2 + x^2 - 4)^2 = 0 \]
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Fig. 4 \( \mathcal{C} : \quad y^2 - x^2 = 7 \)
\[ \mathcal{L} : \quad 4x^2y^4 + x^2(16x^2 - 15)y^6 \\
+ 2(12x^6 + 5x^4 + 143x^2 + 56)y^4 \\
+ (16x^6 + 65x^4 + 2x^4 - 479x^2 - 4)y^2 \\
+ 4x^2(x^4 + 5x^2 + 2)^2 = 0 \]

Fig. 5 \( \mathcal{C} : \quad y^2 - x^2 = 2 \)
\[ \mathcal{L} : \quad 144x^2y^8 + 64x^2(9x^2 + 10)y^6 \\
+ 8(108x^6 + 160x^4 + 257x^2 + 144)y^4 \\
+ 16(36x^8 + 40x^6 - 3x^4 - 114x^2 - 49)y^2 \\
+ 9x^2(4x^4 - 7)^2 = 0 \]

Fig. 6 \( \mathcal{C} : \quad x^2 - 3y^2 = 1 \)
\[ \mathcal{L} : \quad 16x^4y^4 + 4(8x^4 + 70x^2 - 3)y^4 \\
+ (x + 2)(x - 2)(4x^2 - 7)^2 = 0 \]

Fig. 7 \( \mathcal{C} : \quad (x + 2)^2 - y^2 = 1 \)
\[ \mathcal{L} : \quad xy^6 + (3x^3 + 8x^2 + 6x + 6)y^4 \\
+ (3x^2 + 16x^4 + 31x^2 + 20x^2 - 15x - 26)y^2 \\
+ (x + 2)(x^3 + 3x^2 + 2x - 2)^2 = 0 \]
Fig. 8 \( \mathcal{C} : x^2 - y^2 = 2 \)
\[
\mathcal{L} : 16x^2y^8 + 2x^2(32x^2 + 384)y^6 \\
+ 4(24x^6 + 352x^4 + 1762x^2 - 32)y^4 \\
+ 16(4x^8 + 32x^6 - 211x^4 + 400x^2 - 289)y^2 \\
+ x^2(4x^4 - 16x^2 + 17)^2 = 0
\]

Fig. 9 \( \mathcal{C} : (2x + 1)^2 - 4y^2 = 1 \)
\[
\mathcal{L} : 16xy^6 + 8x(6x^2 + 4x + 15)y^4 \\
+ (48x^2 + 64x^4 + 64x^3 - 424x^2 + 81x - 8)y^2 \\
+ (x + 2)(4x^2 - 7x + 1)^2 = 0
\]

Fig. 10-1 \( \mathcal{C} : y = 2x + 2 \)
\[
\mathcal{L} : 125xy^2 + 120y + (5x + 6)(5x + 7)(5x - 1) = 0
\]

Fig. 10-2 \( \mathcal{C} : y = \frac{1}{2} x + \frac{1}{2} \)
\[
\mathcal{L} : 125xy^2 - 120y + (5x - 6)(5x - 7)(5x + 1) = 0
\]
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**Fig. 11** \( C : y = x \)
\[ \mathcal{L} = 4xy^4 + 8x(x^2 + 3)y^2 - 4(6x^2 - 1)y + x(2x^2 + 2x - 1)(2x^2 - 2x - 1) = 0 \]

**Fig. 12** \( C : x^2 + y^2 = 3 \)
\[ \mathcal{L} = 4x^2y^6 + x^4(9x^2 + 56)y^4 + (6x^6 - 27x^4 + 52x^2 - 48)y^2 + (x^2 - 2)^4 = 0 \]

**Fig. 13** \( C : (x - 1)^2 + y^2 = 1 \)
\[ \mathcal{L} = 64x^2y^6 + 64x(3x^3 + 3x^2 + 16x - 4)y^4 + 16x(12x^3 + 32x^2 - 90x^2 - 30x^2 + 133x - 120)y^2 + (x^2 + 4x - 8)^2(8x^2 - 12x + 1)^2 = 0 \]

**Fig. 14** \( C : (x - \frac{1}{2})^2 + y^2 = \left(\frac{1}{2}\right)^2 \)
\[ \mathcal{L} = 64xy^4 + x(89x^2 + 400x + 464)y^2 + (x - 2)(x + 4)^3(5x - 1)^2 = 0 \]
Fig. 15 $\mathcal{C}$ : $(x-1)^2 + (y-2)^2 = 4$

$\mathcal{L}$ : $xy^4 + 2x(2x-1)y^3 + 2(3x^3 + 3x-1)y^2$
$+ 2(2x^3 + 3x^2 + 6x^2 - x - 2)xy$
$+ (x+1)(5x+1)(x^2 + 2x + x - 2) = 0$

Fig. 16 $\mathcal{C}$ : $(x-3)^2 + y^2 = 4$

$\mathcal{L}$ : $xy^4 + 2(13x^3 - 25x^2 + 23x - 5)y^2$
$+ (x-2)(x-1)^2(5x - 7)^2 = 0$

Fig. 17 $\mathcal{C}$ : $(x-1)^2 + y^2 = 2$

$\mathcal{L}$ : $x^2y^5 + 4x(3x^3 - 2x^2 + 14x - 4)y^4$
$+ 4(3x^4 - 18x^4 + 24x^3 - 2x^2 - 8x - 4)y^2$
$+ (x^2 + 2x - 4)^2(2x^2 - 2x^2 - 1)^2 = 0$

Fig. 18 $\mathcal{C}$ : $(x-2)^2 + y^2 = 9$

$\mathcal{L}$ : $36xy^4 + (261x^3 - 1200x^2 + 1216x - 360)y^2$
$+ (x + 2)(3x - 2)^2(5x - 1)^2 = 0$
Plane Algebraic Curves Drawn by the Orthocenter of a Pedal Triangle

Fig. 19-1 \( \mathcal{C} : x^2 + 5y^2 = 1 \)
\[ \mathcal{L} = 144x^2y^4 + (288x^4 + 2200x^2 + 1620)y^2 
+ 9(x + 2)(x - 2)(4x^2 + 1)^2 = 0 \]

Fig. 20-1 \( \mathcal{C} : x^2 + 7y^2 = 1 \)
\[ \mathcal{L} = 324x^2y^4 + (648x^4 + 4185x^2 + 1792)y^2 
+ 4(x + 2)(x - 2)(9x^2 - 2)^2 = 0 \]

Fig. 19-2 \( \mathcal{C} : x^2 + \frac{1}{5}y^2 = 1 \)
\[ \mathcal{L} = 144x^2y^4 + (288x^4 + 2200x^2 + 1620)y^2 
+ 9(x + 2)(x - 2)(4x^2 + 1)^2 = 0 \]

Fig. 20-2 \( \mathcal{C} : x^2 + \frac{1}{7}y^2 = 1 \)
\[ \mathcal{L} = 324x^2y^4 + (648x^4 + 4185x^2 + 1792)y^2 
+ 4(x + 2)(x - 2)(9x^2 - 2)^2 = 0 \]
List 1. A program for drawing a locus

```
--- Specifying statements of the general variables ---
Dim sx As Double, ex As Double
Dim sy As Double, ey As Double
Dim ax As Double, ay As Double
Dim bx As Double, by As Double
Dim cx As Double, cy As Double
Dim xd As Double, yd As Double
Dim xe As Double, ye As Double
Dim xf As Double, yf As Double
Dim k As Integer
Dim ra(4) As Double

--- Drawing the initial figure ---
Private Sub Form_Activate()
    Form1.Line (sx, 0)-(ex, 0)
    Form1.Line (0, sy)-(0, ey)
    For Y = Int(sy) To Int(ey)
        Form1.Line (-0.02, Y)-(0.02, Y)
    Next Y
    Form1.DrawWidth = 2
    Form1.Line (sx, sx)-(ex, ex), QBColor(9)
    Form1.Line (bx, by)-(cx, cy)
    Form1.DrawMode = vbNotXorPen
    Form1.Line (ax, ay)-(bx, by)
    Form1.Line (cx, cy)-(ax, ay)
    Form1.DrawWidth = 1
    Anth_triangle ax, ay, bx, by, cx, cy, 2
End Sub

--- Setting the form, the coordinate axes and the initial values of the coordinates of each vertex of the triangle ---
Private Sub Form_Load()
    Form1.Top = 100
    Form1.Left = 2000
    Form1.Width = 0.5 * Screen.Width
    Form1.Height = 0.9 * Screen.Height
    sx = -2
    ex = 2
    wx = ex - sx
    wy = wx * Form1.ScaleHeight / Form1.ScaleWidth
    sy = -0.5 * wy
    ey = 0.5 * wy
```
Plane Algebraic Curves Drawn by the Orthocenter of a Pedal Triangle

```vbnet
Form1.Scale (sx, ey)-(ex, sy)
Form1.BackColor = vbWhite
Form1.DrawWidth = 1
ax = 0.6: ay = 0.6
bx = -1
by = 0
cx = 1
cy = 0
k = 1
End Sub

' Action corresponding to the left button of mouse
Private Sub Form_MouseDown(Button As Integer, Shift As Integer, x As Single, Y As Single)
    If k = 2 Then
        k = 1
        Exit Sub
    End If
    k = 2
    Disp x
End Sub

' Action corresponding to the movement of mouse
Private Sub Form_MouseMove(Button As Integer, Shift As Integer, X As Single, Y As Single)
    If k = 1 Then Exit Sub
    Disp x
End Sub

' Presenting of the orthocenter of a triangle
Private Sub Orthocenter(jxa, jya, jxb, jyb, jxc, jyc, pcl)
    xa = jxa: ya = jya
    xb = jxb: yb = jyb
    xc = jxc: yc = jyc
    ux = xb - xa: uy = yb - ya
    vx = xa - xc: vy = ya - yc
    wx = xc - xb: wy = yc - yb
    a = Sqr(wx ^ 2 + wy ^ 2)
    b = Sqr(vx ^ 2 + vy ^ 2)
    c = Sqr(ux ^ 2 + uy ^ 2)
    ah = b ^ 2 + c ^ 2 - a ^ 2
    bh = c ^ 2 + a ^ 2 - b ^ 2
    ch = a ^ 2 + b ^ 2 - c ^ 2
    If ah >= 0 And bh >= 0 And ch >= 0 And a > 10 ^ (-4) And b > 10 ^ (-4) And c > 10 ^ (-4)
    p1 = xb * yc - xc * yb: q1 = -wy * ya - wx * xa
    p2 = xc * ya - xa * yc: q2 = -vy * yb - vx * xb
    p3 = xa * yb - xb * ya: q3 = -uy * yc - ux * xc
```
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\[
\begin{align*}
xd &= (p1 * wy - q1 * wx) / a^2 \\
yd &= (-p1 * wx - q1 * wy) / a^2 \\
xe &= (p2 * vy - q2 * vx) / b^2 \\
ye &= (-p2 * vx - q2 * vy) / b^2 \\
xf &= (p3 * uy - q3 * ux) / c^2 \\
yf &= (-p3 * ux - q3 * uy) / c^2 \\
xh &= (q2 * wy - q1 * vy) / (wx * vy - wy * vx) \\
yh &= (q1 * vx - q2 * wx) / (wx * vy - wy * vx)
\end{align*}
\]

Form1.DrawStyle = 2
Form1.Line(xa, ya)-(xd, yd), vbBlue
Form1.Line(xb, yb)-(xe, ye), vbBlue
Form1.Line(xc, yc)-(xf, yf), vbBlue
Form1.DrawStyle = 0
Form1.DrawWidth = 4
Form1.DrawMode = vbCopyPen
Form1.PSet(xh, yh), QBColor(pcl)
Form1.DrawMode = vbNotXorPen
Form1.DrawWidth = 1
Else
If bh < 0 Then
  dm = xa: xa = xb: xb = dm
  dm = ya: ya = yb: yb = dm
End if
If ch < 0 Then
  dm = xa: xa = xc: xc = dm
  dm = ya: ya = yc: yc = dm
End if

ux = xb - xa: uy = yb - ya
vx = xa - xc: vy = ya - yc
wx = xc - xb: wy = yc - yb
d = wx * vy - wy * vx
If Abs(d) > 0.001 Then
  a = Sqr(wx^2 + wy^2)
  b = Sqr(vx^2 + vy^2)
  c = Sqr(ux^2 + uy^2)
  p1 = xb * yc - xc * yb: q1 = -wy * ya - wx * xa
  q2 = -vy * yb - vx * xb
  xd = (p1 * wy - q1 * wx) / a^2
  yd = (-p1 * wx - q1 * wy) / a^2
  xh = (q2 * wy - q1 * vy) / d
  yh = (q1 * vx - q2 * wx) / d
H_Prolong xb, yb, xa, ya
H_Prolong xc, yc, xa, ya
Form1.DrawLine = 2
Form1.DrawLine(xh, yh)-(xd, yd), vbBlue
Form1.DrawLine(xb, yb)-(xh, yh), vbBlue
Form1.DrawLine(xc, yc)-(xh, yh), vbBlue
Form1.DrawLine = 0
Form1.DrawWidth = 4
Form1.DrawMode = vbCopyPen
Form1.PSet(xh, yh), QBColor(pcl)
Form1.DrawMode = vbNotXorPen
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Form1.DrawWidth = 1
End If
End If
End Sub

' Extending of two sides adjacent at the vertex with an obtuse angle

Private Sub H_Prolong(x1, y1, x2, y2)
    Form1.DrawWidth = 1
    l = Sqr((x1 - x2) ^ 2 + (y1 - y2) ^ 2)
    ra(1) = Sqr((x1 - sx) ^ 2 + (y1 - sy) ^ 2)
    ra(2) = Sqr((x1 - ex) ^ 2 + (y1 - ey) ^ 2)
    ra(3) = Sqr((x1 - ex) ^ 2 + (y1 - ey) ^ 2)
    ra(4) = Sqr((x1 - sx) ^ 2 + (y1 - ey) ^ 2)
    r = ra(1)
    For i = 2 To 4
        If ra(i) > r Then r = ra(i)
    Next i
    p = r / l
    tx = (1 - p) * x1 + p * x2
    ty = (1 - p) * y1 + p * y2
    Form1.DrawStyle = 2
    Form1.Line(x2, y2) - (tx, ty), QBColor(9)
    Form1.DrawStyle = 0
End Sub

' Presenting of a pedal triangle

Private Sub Anth_triangle(xa, ya, xb, yb, xc, yc, col)
    ux = xb - xa: uy = yb - ya
    vx = xa - xc: vy = ya - yc
    wx = xc - xb: wy = yc - yb
    a = Sqr(wx * 2 + wy * 2)
    b = Sqr(vx * 2 + vy * 2)
    c = Sqr(ux * 2 + uy * 2)
    If a > 10 ^ (-4) And b > 10 ^ (-4) And c > 10 ^ (-4) Then
        ah = b ^ 2 + c ^ 2 - a ^ 2
        bh = c ^ 2 + a ^ 2 - b ^ 2
        ch = a ^ 2 + b ^ 2 - c ^ 2
        p1 = xb * yc - xc * yb: q1 = -wy * ya - wx * xa
        p2 = xc * ya - xa * yc: q2 = -vy * yb - vx * xb
        p3 = xa * yb - xb * ya: q3 = -uy * yc - ux * xc
        xd = (p1 * wy - q1 * wx) / a ^ 2
        yd = (-p1 * wx - q1 * wy) / a ^ 2
        xe = (p2 * vy - q2 * vx) / b ^ 2
        ye = (-p2 * vx - q2 * vy) / b ^ 2
        xf = (p3 * uy - q3 * ux) / c ^ 2
        yf = (-p3 * ux - q3 * uy) / c ^ 2
        Form1.DrawWidth = 5
Form1.PSet (xd, yd), QBColor(col)
Form1.PSet (xe, ye), QBColor(col)
Form1.PSet (xf, yf), QBColor(col)
Form1.DrawWidth = 2
Form1.Line (xd, yd)–(xe, ye), QBColor(col)
Form1.Line (xe, ye)–(xf, yf), QBColor(col)
Form1.Line (xf, yf)–(xd, yd), QBColor(col)
Form1.DrawWidth = 1
Orthocenter xd, yd, xe, ye, xf, yf, 12
End If
End Sub

Private Sub Ext_Click()
For x = ex To 40 * ex Step 0.1
Disp x
Next x
For x = 40 * sx To sx Step 0.1
Disp x
Next x
For x = 2 * sx To 2 * ex Step 0.001
Disp x
Next x
Disp 1.5
End Sub

Private Sub Disp(x)
Form1.DrawWidth = 2
Form1.Line (ax, ay)–(bx, by)
Form1.Line (cx, cy)–(ax, ay)
Y = x
Form1.Line (x, Y)–(bx, by)
Form1.Line (cx, cy)–(x, Y)
Form1.DrawWidth = 1
Anth_triangle ax, ay, bx, by, cx, cy, 2
Anth_triangle x, Y, bx, by, cx, cy, 2
ax = x: ay = Y
End Sub