A Note on the Tensor Products of Algebras over a Field

Kazunori Fujita

Introduction. All rings considered in this paper are assumed to be commutative with identity, and all ring homomorphisms are unital. Throughout, $k$ stands for a field. A ring $R$ is said to be a Hilbert ring if it satisfies one of the following equivalent conditions.

1. Every prime ideal of $R$ is an intersection of maximal ideals.
2. If $M$ is a maximal ideal in $R[X]$, then $M \cap R$ is a maximal ideal.

Let $K$ and $L$ be extension fields of $k$. If $\text{tr.deg}_k K \geq \text{tr.deg}_k L = n < \infty$, then $K \otimes_k L$ is a Hilbert ring in which every maximal ideal has height $n$ ([16], Theorem 5). This result is generalized in [6], [11], [12] and [15]. In the article [5], the following question was given.

Question: Let $A$ and $B$ be two Hilbert $k$-algebras such that $A \otimes_k B$ is noetherian. Is $A \otimes_k B$ a Hilbert ring? Is it equicodimensional?

In this paper, we show some conditions of Hilbert $k$-algebras $A$ and $B$ when $A \otimes_k B$ is a Hilbert ring. Any unreferenced material is standard, as in [4] and [10].

Lemma 1. If $R$ is a Hilbert ring, then $R[X_1, \ldots, X_n]$ is a Hilbert ring for each positive integer $n$

Proof. See [3, (31.8)]

Proposition 1. Let $A$ and $B$ be two $k$-algebras. If $A$ is finitely generated over $k$, and $B$ is a Hilbert ring, then $A \otimes_k B$ is a Hilbert ring.

Proof. Let $A = k[a_1, \ldots, a_n]$, and $\varphi : k[X_1, \ldots, X_n] \to A$ be a $k$-algebra homomorphism such that $\varphi(X_i) = a_i$. Since $B$ is a Hilbert ring, $k[X_1, \ldots, X_n] \otimes_k B \cong B[X_1, \ldots, X_n]$ is a Hilbert ring by Lemma 1. Therefore its homomorphic image $A \otimes_k B$ is a Hilbert ring.

Lemma 2. Let $K$ be a field of uncountable cardinality, $A$ be a countably generated $K$-algebra. Then $A$ is a Hilbert ring.

Proof. Let $\mathcal{K}$ be the algebraic closure of $K$. Then $\mathcal{K}[\{X_n \mid n \in \mathbb{N}\}]$ is a Hilbert ring by
Theorem in [9, p.407]. $\mathcal{K}[\{X_n \mid n \in \mathbb{N}\}]$ is integral over $B = \mathcal{K}[\{X_n \mid n \in \mathbb{N}\}]$, so $B$ is a Hilbert ring. $A$ is a homomorphic image of $B$, hence $A$ is a Hilbert ring.

**Proposition 2.** Let $k$ be a field of countable cardinality, $K$ be a field of uncountable cardinality containing $k$. If $A$ is a countably generated $k$-algebra, and if $B$ is a countably generated $K$-algebra. Then $A \otimes_k B$ is a Hilbert ring.

**Proof.** $A \otimes_k B$ is a countably generated $K$-algebra. Therefore $A \otimes_k B$ is a Hilbert ring by Lemma 2.

**Example 1.** $\mathbb{Q}(X_1, X_2, \cdots, X_n) \otimes_{\mathbb{Q}[Y_1, Y_2, Y_3, \cdots]} \mathbb{Q}[Y_1, Y_2, Y_3, \cdots]$ is a Hilbert ring by Proposition 2.

**Example 2.** $\mathbb{Q}[X_1, X_2, X_3, \cdots] \otimes_{\mathbb{Q}} \mathbb{R}[X_1, X_2, X_3, \cdots]$ is a Hilbert ring by Lemma 2, however $\mathbb{Q}[X_1, X_2, X_3, \cdots]$ is not a Hilbert ring.

**Definitions.** ([17], p.394) Let $A$ be an integral domain over $k$. $A$ is an $AF$-domain (altitude formula) if

$$\text{ht}(P) + \text{tr.deg}_k(A/P) = \text{tr.deg}_k(A)$$

for each prime ideal $P$ of $A$.

**Definitions.** ([17], p.395) Let $R$ be a $k$-algebra, $p$ a prime ideal of $R$ and $0 \leq d \leq s$ be integers. Set

$$\Delta(s, d, p) := \text{ht}(p[X_1, \cdots, X_s]) + \min(s, d + \text{tr.deg}_k(R/p))$$

$$D(s, d, R) := \max\{\Delta(s, d, p) \mid p \in \text{Spec}(R)\}$$

**Lemma 3.** ([17], Theorem 3.7). Let $A$ be an AF-domain with $t = \text{tr.deg}_k(A)$ and $d = \dim(A)$. Let $R$ be any $k$-algebra. Then

$$\dim(A \otimes_k R) = D(t, d, R)$$

$$= \max\{\text{ht}(pR[X_1, \cdots, X_t]) + \min(t, d + \text{tr.deg}_k(R/p)) \mid p \in \text{Spec}(R)\}$$

**Lemma 4.** ([17], Corollary 4.2). Let $A$ be an integral domain containing $k$. If $A$ is noetherian, or is a Prüfer domain (e.g., a valuation domain), or AF-domain, then

$$\dim(A \otimes_k A) = \dim(A) + \text{tr.deg}_k(A).$$
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Let $R$ be a ring, and let $U$ be the multiplicatively closed subset of monic polynomials of $R[X]$. Then the ring $U^{-1}R[X]$ is denoted by $R\langle X \rangle$. Some basic properties of rings of this type are studied in [1], [14]. If $R$ is a noetherian ring, then $R\langle X \rangle$ is a Hilbert ring ([2], [13]).

**Example 3.** Let $(V, M)$ be a discrete rank-one valuation domain containing $k$. Then

1. $(V\langle X \rangle)[Y]$ is a Hilbert domain such that every maximal ideal is of height 2.
2. $k(Y) \otimes_k V\langle X \rangle$ is a Hilbert domain.

**Proof.** (1) $V\langle X \rangle$ is a 1-dimensional noetherian Hilbert domain, so that $(V\langle X \rangle)[Y]$ is a Hilbert domain and for every maximal ideal $M$ of $(V\langle X \rangle)[Y]$, $\text{ht}(M) = 2$.

(2) Lemma 3 implies that $\dim (k(Y) \otimes_k V\langle X \rangle) = 2$. Let $S = k[Y] - \{0\}$. Then it can be shown that $S^{-1}(V\langle X \rangle)[Y]/P$ is not semi-local for each height one prime ideal $P$ of $S^{-1}(V\langle X \rangle)[Y]$. Thus $k(Y) \otimes_k V\langle X \rangle$ is a Hilbert domain.

**Lemma 5** ([4], p.106). Let $R$ be a noetherian ring, and let $s$ be a non-nilpotent element of $\text{rad}(R)$. Then $R_s$ is a Hilbert ring.

**Example 4.** Let $(V, M)$ be a discrete rank-one valuation domain containing $k$ such that $\text{tr.deg}_k V = n < \infty$. Then $\dim (V\langle X \rangle \otimes_k V\langle X \rangle) = n + 2$ by Lemma 4. Is $V\langle X \rangle \otimes_k V\langle X \rangle$ is a Hilbert ring? Let $B = (1+YV[Y])^{-1}V[Y]$, where $Y = 1/X$. Then $V\langle X \rangle = B[1/Y]$, $B = V[Y]_{(M, Y)}$, so that $Y$ is contained in $\text{rad}(B)$ by Proposition 1.4 in [8]. If $B \otimes_k B$ is noetherian, and if $Y \otimes 1$, $1 \otimes Y \in \text{rad}(B \otimes_k B)$, then $V\langle X \rangle \otimes_k V\langle X \rangle$ is a Hilbert ring by Lemma 5.

**References**


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[7] H. A. Khashan and H. Al-egeh, Conditions under which $R(x)$ and $R\langle x \rangle$ are almost $\mathbb{Q}$-Rings, Archivum Mathematicum (BRNO), Tomus 43 (2007), 231 – 236.


