A Locus of the Orthocenter of a Triangle

—Instruction in Geometry by a Moving Locus on a Computer—

by

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Abstract

An effective use of a computer makes mathematics classes much more interesting and motivate many students to learn more. In this paper we present a program just like a computer game for drawing a locus of the orthocenter of a triangle with two vertices fixed when one moves the third vertex along a curve. A locus moving on a display provides an active teaching material of elementary geometry for students.

§ 1. Introduction

In elementary geometry we have five significant notions for a triangle; that is, the center of gravity, the center of an inscribed circle, the center of an escribed circle, the circumcenter and the orthocenter of a triangle. Which curve is drawn as a locus of such a point on the plane when the third vertex of a triangle with two vertices fixed moves under a certain condition?

In this study we limit ourselves to the case of a locus of the orthocenter of a triangle with two vertices fixed where the third vertex moves along a distinguished curve on the plane. Then our main concern is to find all the types of remarkable curves with a simple expression as a locus of the orthocenter of a triangle.

We intend to write a program for drawing a locus of the orthocenter on a display
and to apply it to elementary geometry classes as an activity of experimental computer use.

As for the present paper, the first author gave a general idea for drawing a locus of the orthocenter of a triangle on a display and made some preliminary programs for teaching elementary geometry like a computer game. After a while, he noticed that Inosako [3] referred to a similar teaching plan for computer use in a senior high school. The second author wrote the programs for all of the figures under the direction of the first author. The last author joined the discussions and arranged the results for publishing.

§ 2. A program for drawing a locus

We divide our operation into two parts, drawing and printing, due to the circumstances of our computer machines.

PART ONE: To draw a locus of the orthocenter of a triangle.
Our program is written in Visual Basic Ver. 6.0 of Microsoft Corporation and consists of the following six steps:

Step 1. Two vertices $B$, $C$ of a triangle are fixed on a display beforehand and a curve $\mathcal{C}$ (blue) along which the third vertex $A$ of a triangle will move is also fixed.

Step 2. One can arbitrarily select a point $A$ on the curve $\mathcal{C}$ for the initial point by the mouse. Then the point is marked in black and each side of $\triangle ABC$ is drawn as a solid black line-segment.

Step 3. The orthocenter $H$ of $\triangle ABC$ is plotted in red.

Step 4. Each of the perpendiculars from the vertices $A$, $B$, $C$ to the opposite sides is drawn with a dotted blue line.

Step 5. When $\triangle ABC$ is obtuse, each of two sides at the adjacent vertex with the obtuse angle is extended with a dotted green line.

Step 6. When one moves the point $A$ continuously along the curve $\mathcal{C}$, the orthocenter $H$ of $\triangle ABC$ continuously draws a locus $\mathcal{L}$ with a solid red line.
PART TWO: To process a bitmap file (bmp) by \LaTeX to exhibit it on another display and to print it.

OUTLINE of the PROGRAM. Let \( B \left( -\sqrt{1-0.5^2}, -0.5 \right) \), \( C \left( \sqrt{1-0.5^2}, -0.5 \right) \) be two vertices of a triangle on the \( xy \)-plane and let \( \mathcal{C} \) be the circle given by the equation \( x^2 + y^2 = 1 \). When a point \( A \) on \( \mathcal{C} \) is selected by the mouse, put its coordinates by \( (x, y) \). Then the orthocenter \( H (u, v) \) of \( \triangle ABC \) is given by

\[
\begin{cases}
  u = x, \\
  v = \frac{-2x - y^2 + 1}{2y + 1}.
\end{cases}
\]

The program for drawing a locus \( \mathcal{L} \) of \( H \) is given in List 1. In this case one will have a circle \( \mathcal{L} : x^2 + (y+1)^2 = 1 \) on a display (Fig. 1).

We keep the same notations as in this section throughout the paper.

§ 3. Curves obtained as a locus of the orthocenter

As a locus of the orthocenter of a triangle we have various classical curves containing all the conics such as listed in a text [2; pp. 520-521]. This fact indicates that each of those curves arises under a usual situation. In this section we list the typical curves obtained as a locus of the orthocenter of a triangle. Each of the following Propositions is established by an example.

**Proposition 1.** A parabola can be obtained as a locus of the orthocenter of a triangle when the third vertex moves along a straight line.

**Example 1.** (Fig. 2) Let \( B (-1, 0) \), \( C (1, 0) \) and \( \mathcal{C} : y = -2 \). Then we have a parabola \( y = \frac{x^2}{2} - \frac{1}{2} \) as the locus \( \mathcal{L} \).

**Proposition 2.** A hyperbola can be obtained as a locus of the orthocenter of a triangle when the third vertex moves along a straight line.
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**Example 2.1. (Fig. 3)** Let \(B(-1, 0), \ C(1, 0)\) and \(C: \ y=x\). Then we have a hyperabola \(x^2+xy-1=0\) as the locus \(L\).

**Example 2.2. (Fig. 4)** Let \(B(-1, 0), \ C(1, 0)\) and \(C: \ y=\sqrt{3}x+1\). Then we have a hyperabola \(x^2+\sqrt{3}xy+y-1=0\) as the locus \(L\).

To verify the result, take a new coordinate system of the plane such as \(x=X-\frac{\sqrt{3}}{3}\), \(y=Y+\frac{2}{3}\) and rotate the curve \(L\) by an angle of 30° about the new origin \(O\) in the \(XY\)-plane. Then we have a normal form of hyperabola

\[
\frac{X^2}{\left(\frac{2}{3}\right)^2} - \frac{Y^2}{\left(\frac{2}{3}\sqrt{3}\right)^2} = 1.
\]

**Proposition 3.** A rectangular hyperabola can be obtained as a locus of the orthocenter of a triangle when the third vertex moves along a cubic curve.

**Example 3. (Fig. 5)** Let \(B(-1, 0), \ C(1, 0)\) and \(C: \ y=-x^3+x\). Then we have a rectangular hyperabola \(xy=1\) as the locus \(L\).

**Proposition 4.** A parabola can be obtained as a locus of the orthocenter of a triangle when the third vertex moves along a curve given by a quadratic fractional function.

**Example 4. (Fig. 6)** Let \(B(-1, 0), \ C(1, 0)\) and \(C: \ y=\frac{1}{x^2}-1\). Then we have a parabola \(y=x^2\) as the locus \(L\).

**Proposition 5.** An ellipse can be obtained as a locus of the orthocenter of a triangle when the third vertex moves along a folium of Descartes.

**Example 5. (Fig. 7)** Let \(B(0, 0), \ C(1.5, 1.5)\) and \(C: \ x^3-3xy+y^3=0\) (a folium of Descartes). Then we have an ellipse as the locus \(L: 4x^2+4xy+4y^2-6x-6y-9=0\).

To verify the result, rotate the curve \(C\) by an angle of \((-45°)\) about the origin. Then we have the equation of the curve \(C'\) as

\[
C': \ y^2 = \frac{x^2(3-\sqrt{2}x)}{3\sqrt{2}x+3}.
\]
By the rotation the vertex $C$ moves to the point $C' \left( \frac{3}{2}, \sqrt{2}, 0 \right)$. When a point $A'$ moves along $\mathcal{C}'$ the locus $\mathcal{L}'$ of the orthocenter $H'$ of $\triangle A'BC'$ is given by a normal form of ellipse
\[
\frac{(x - \frac{3}{2})^2}{(\sqrt{2})^2} + \frac{y^2}{(\sqrt{6})^2} = 1.
\]

**Proposition 6.** An ellipse can be obtained as a locus of the orthocenter of a triangle when the third vertex moves along a strophoid.

**Example 6.** (Fig. 8) Let $B(0, 0)$, $C(1, 0)$ and $\mathcal{C}: y^2 = \frac{x^2(1-x)}{a^2(1+x)}$ for $a > 0$.

Then we have an ellipse $x^2 + \frac{y^2}{a^2} = 1$ as the locus $\mathcal{L}$. Fig. 8 shows the locus $\mathcal{L}: x^2 + \frac{y^2}{4} = 1$ for $a = \frac{1}{2}$.

**Proposition 7.** A strophoid can be obtained as a locus of the orthocenter of a triangle when the third vertex moves along a circle.

**Example 7.** (Fig. 9) Let $B(0, 0)$, $C(1, 0)$ and $\mathcal{C}: x^2 + y^2 = 1$. Then we have a strophoid
\[
y^2 = x^2 \frac{1-x}{1+x}
\]
as the locus $\mathcal{L}$.

**Proposition 8.** A cissoid can be obtained as a locus of the orthocenter of a triangle when the third vertex moves along a strophoid.

**Example 8.** (Fig. 10) Let $B(-1, 0)$, $C(0, 0)$ and $\mathcal{C}: y^2 = x^2 \frac{1-x}{1+x}$ (a strophoid). Then we have a cissoid
\[
y^2 = \frac{(1+x)^3}{1-x}
\]
as the locus $\mathcal{L}$. 

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**Proposition 9.** A witch of Agnesi can be obtained as a locus of the orthocenter of a triangle when the third vertex moves along curve given by a quartic equation in $x$ and $y$.

**Example 9. (Fig. 11)** Let $B(-1, 0)$, $C(1, 0)$ and $\mathcal{C}: y^2 = x(1-x)(1+x)^2$. Then we have a witch of Agnesi

$$y^2 = \frac{1-x}{x}$$

as the locus $\mathcal{L}$.

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**§ 4. Remarks**

**Remark 1.** The transformation $T: A \mapsto H$ is a bijection on the Riemannian sphere $S = \mathbb{R}^2 \cup \{\infty\}$ and $T^{-1} = T$. In fact, a point $H$ is the orthocenter of $\triangle ABC$ if and only if a point $A$ is the orthocenter of $\triangle HBC$.

Therefore, it is known that a folium of Descartes can be obtained as a locus of the orthocenter of $\triangle ABC$, when a point $A$ moves along an ellipse from Proposition 5 (Fig. 7).

**Remark 2.** When the third vertex of triangle moves along a straight line $\mathcal{C}$, we have a parabola or a hyperabola or a perpendicular to the side $BC$ through the endpoint as the locus $\mathcal{L}$.

**Remark 3.** The curve $\mathcal{L}$ shown in Fig. 12 appears among many experiments of drawing a locus by a computer. It is beautiful and mysterious to us. What secrets do it possess?
References


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Fig. 1  $C : x^2 + y^2 = 1$
$L : x^2 + (y+1)^2 = 1$

Fig. 2  $C : y = -2$
$L : y = \frac{x^2}{2} - \frac{1}{2}$

Fig. 3  $C : y = x$
$L : x^2 + xy - 1 = 0$

Fig. 4  $C : y = \sqrt{3}x + 1$
$L : x^2 + \sqrt{3}xy + y - 1 = 0$

Fig. 5  $C : y = -x^2 + x$
$L : xy = 1$

Fig. 6  $C : y = \frac{1}{x^2} - 1$
$L : y = x^3$
Fig. 7 \( C : x^3 - 3xy + y^2 = 0 \)
\[ L : 4x^2 + 4xy + 4y^2 - 6x - 6y - 9 = 0 \]

Fig. 8 \( C : y^2 = \frac{x^2(1+x)}{4(1-x)} \)
\[ L : x^2 + \frac{y^2}{4} = 1 \]

Fig. 9 \( C : x^2 + y^2 = 1 \)
\[ L : y^2 = x^2 \frac{1-x}{1+x} \]

Fig. 10 \( C : y^2 = x^3 \frac{1-x}{1+x} \)
\[ L : y^2 = \frac{(1+x)^3}{1-x} \]

Fig. 11 \( C : y^2 = x(1-x)(1+x)^2 \)
\[ L : y^2 = \frac{1-x}{x} \]

Fig. 12 \( C : x^2 + (y - \frac{5}{2})^2 = 4 \)
\[ L : (4-x^2)y^2 = (1-x^2 - \frac{5}{2}y)^2 \]
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List 1. A program for drawing a locus

Dim sx As Double, ex As Double
Dim sy As Double, ey As Double
Dim ax As Double, ay As Double
Dim bx As Double, by As Double
Dim cx As Double, cy As Double
Dim k As Integer
Dim ra(4) As Double

'-------------------------------------------------
' Drawing the initial triangle
'-------------------------------------------------
Private Sub Form_Activate()
  Form1.AutoRedraw = True
  Form1.Circle (0, 0), 1
  Form1.Line (bx, by)-(cx, cy)
  Form1.DrawMode = vbNotXorPen
  Form1.Line (ax, ay)-(bx, by)
  Form1.Line (cx, cy)-(ax, ay)
  Orthocenter ax, ay, bx, by, cx, cy
End Sub

'-------------------------------------------------
' Setting the coordinate axes and the initial values of
' the coordinates of each vertex of a triangle
'-------------------------------------------------
Private Sub Form_Load()
  sx = -2.2
  ex = 2.2
  wx = ex - sx
  wy = wx * Form1.ScaleHeight / Form1.ScaleWidth
  sy = -0.6 * wy
  ey = 0.4 * wy
  Form1.Scale (sx, ey)-(ex, sy)
  Form1.BackColor = vbWhite
  Form1.DrawWidth = 1
  ax = 0.6: ay = 0.8
  by = -0.5
  bx = Sqr(1 - by ^ 2)
  cy = -0.5
  cx = Sqr(1 - cy ^ 2)
  k = 1
End Sub

'-------------------------------------------------
' Action corresponding to the left button of mouse
'-------------------------------------------------
Private Sub Form_MouseDown(Button As Integer, Shift As Integer, X As Single, Y As Single)
  If k = 2 Then
    k = 1
  Exit Sub
  End Sub
End If

k = 2

If d1 < 0.5 Then
  Form1.Line (ax, ay)-(bx, by)
  Form1.Line (cx, cy)-(ax, ay)
  r = Sqr(X^2 + Y^2)
  X = X / r; Y = Y / r
  Form1.Line (X, Y)-(bx, by)
  Form1.Line (cx, cy)-(X, Y)
  Orthocenter ax, ay, bx, by, cx, cy
  Orthocenter X, Y, bx, by, cx, cy
  ax = X: ay = Y
End If
End Sub

' Action corresponding to the movement of mouse

Private Sub Form_MouseMove(Button As Integer, Shift As Integer, X As Single, Y As Single)
  If k = 1 Then Exit Sub
  d1 = Sqr((ax-X)^2 + (ay-Y)^2)
  If d1 < 0.5 Then
    Form1.Line (ax, ay)-(bx, by)
    Form1.Line (cx, cy)-(ax, ay)
    r = Sqr(X^2 + Y^2)
    X = X / r; Y = Y / r
    Form1.Line (X, Y)-(bx, by)
    Form1.Line (cx, cy)-(X, Y)
    Orthocenter ax, ay, bx, by, cx, cy
    Orthocenter X, Y, bx, by, cx, cy
    ax = X: ay = Y
  End If
End Sub

' Presentation of the orthocenter of a triangle

Private Sub Orthocenter(jxa, jya, jxb, jyb, jxc, jyc)
  xa = jxa: ya = jya
  xb = jxb: yb = jyb
  xc = jxc: yc = jyc
  ux = xb - xa: uy = yb - ya
  vx = xa - xc: vy = ya - yc
  wx = xc - xb: wy = yc - yb
  a = Sqr(wx^2 + wy^2)
  b = Sqr(vx^2 + vy^2)
  c = Sqr(ux^2 + uy^2)
  ah = b^2 + c^2 - a^2
  bh = c^2 + a^2 - b^2
  ch = a^2 + b^2 - c^2
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'------------------------------------------
' Procedure for an acute triangle
'------------------------------------------
If ah >= 0 And bh >= 0 And ch >= 0 Then
  p1 = xb * yc - xc * yb: q1 = -wy * ya - wx * xa
  p2 = xc * ya - xa * yc: q2 = -vy * yb - vx * xb
  p3 = xa * yb - xb * ya: q3 = -uy * yc - ux * xc
  xd = (-p1 * wx - q1 * wy) / a ^ 2
  yd = (-p1 * wx - q1 * wy) / a ^ 2
  xe = (p2 * vy - q2 * vx) / b ^ 2
  ye = (-p2 * vx - q2 * vy) / b ^ 2
  xf = (p3 * uy - q3 * ux) / c ^ 2
  yf = (-p3 * ux - q3 * uy) / c ^ 2
  xh = (q2 * vy - q1 * vy) / (wx * vy - wy * vx)
  yh = (q1 * vx - q2 * wx) / (wx * vy - wy * vx)
Form1.DrawStyle = 2
Form1.Line (xa, ya)-(xd, yd)
Form1.Line (xb, yb)-(xe, ye)
Form1.Line (xc, yc)-(xf, yf)
Form1.DrawStyle = 0
Form1.DrawWidth = 5
Form1.DrawMode = vbCopyPen
Form1.PSet (xh, yh), vbRed
Form1.DrawMode = vbNotXorPen
Form1.DrawWidth = 1

'------------------------------------------
' Procedure for an obtuse triangle
'------------------------------------------
Else
  If bh < 0 Then
    dm = xa: xa = xb: xb = dm
    dm = ya: ya = yb: yb = dm
  End If

  If ch < 0 Then
    dm = xa: xa = xc: xc = dm
    dm = ya: ya = yc: yc = dm
  End If

  ux = xb - xa: uy = yb - ya
  vx = xa - xc: vy = ya - yc
  wx = xc - xb: wy = yc - yb
  d = wx * vy - wy * vx
  If Abs(d) > 0.001 Then
    a = Sqr(wx ^ 2 + wy ^ 2)
    b = Sqr(vx ^ 2 + vy ^ 2)
    c = Sqr(ux ^ 2 + uy ^ 2)
    p1 = xb * yc - xc * yb: q1 = -wy * ya - wx * xa
    q2 = -vy * yb - vx * xb
    xd = (p1 * wy - q1 * wx) / a ^ 2
    yd = (-p1 * wx - q1 * wy) / a ^ 2
  End If

'------------------------------------------
xh = (q2 * wy - q1 * vy) / d
yh = (q1 * vx - q2 * wx) / d
H_Prolong xb, yb, xa, ya
H_Prolong xc, yc, xa, ya
Form1.DrawStyle = 2
Form1.Line (xh, yh)-(xd, yd), vbBlue
Form1.Line (xb, yb)-(xh, yh), vbBlue
Form1-Line (xc, yc)-(xh, yh), vbBlue
Form1.DrawStyle = 0
Form1.DrawWidth = 5
Form1.DrawMode = vbCopyPen
Form1.PSet (xh, yh), vbRed
Form1.DrawMode = vbNotXorPen
Form1.DrawWidth = 1
End If
End If
End Sub

' Extension of two sides at the adjacent vertex with an obtuse angle

Private Sub H_Prolong(x1, y1, x2, y2)
L = Sqr((x1 - x2) ^ 2 + (y1 - y2) ^ 2)
ra(1) = Sqr((x1 - sx) ^ 2 + (y1 - sy) ^ 2)
ra(2) = Sqr((x1 - ex) ^ 2 + (y1 - sy) ^ 2)
ra(3) = Sqr((x1 - ex) ^ 2 + (y1 - ey) ^ 2)
ra(4) = Sqr((x1 - sx) ^ 2 + (y1 - ey) ^ 2)

r = ra(1)
For i = 2 To 4
If ra(i) > r Then r = ra(i)
Next i
p = r / L
tx = (1 - p) * x1 + p * x2
ty = (1 - p) * y1 + p * y2
Form1.DrawStyle = 2
Form1.Line (x2, y2)-(tx, ty), Q8Color(2)
Form1.DrawStyle = 0
End Sub

' Saving the picture in the bitmap file

Private Sub Form_Load()
Filnm = "C:\Bmp_File\OrthoC1.8mp"
SavePicture Form1.Image, Filnm
End Sub