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A Triangle with Three Distinguished Collinear Points

— Instruction of Geometry by Use of a Drawing Game on a Display —

by

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Abstract

An effective use of a computer makes mathematics classes much more interesting and motivates many students to learn.

In elementary geometry we have five significant notions for a triangle; that is, the center of gravity, the incenter, the excenter, the circumcenter and the orthocenter of a triangle. Concerning those five points we have a natural problem. Determine a triangle on the plane with one of the following conditions :

(1) Three of those five points lie in a common straight line,

(2) A vertex lies on the straight line through two of those five points.

In this paper we present a drawing game for a student to find a solution to the above problem with fun, and obtain some results from our various experiments.

§ 1. Introduction

In the sequel to [**3**] our study aims to develop a drawing game on a display as a teaching material for elementary geometry classes with activities using computers.

In elementary geometry we have five significant notions for a triangle ; that is, the center of gravity, the incenter, the excenter, the circumcenter and the orthocenter of a triangle. Concerning those points we have a natural problem. Determine a triangle on the plane with one of the following conditions :

(1) Three of those five points lie in a common straight line,
(2) A vertex lies on the straight line through two of those five points.

In this paper we present a drawing game for a student to find a solution to the above problem with fun, and obtain some results from our various experiments. Some of them are already known. In fact, "Mathematical Discovery" due to G. Polya [**6**] indicates the spirit of our research in mathematics education. For terminology of geometry throughout the paper, consult [**1**], [**2**] and [**5**].

As for the present paper, the first author gave a general idea for drawing a triangle and related figures on a display and made some preliminary programs for teaching materials in elementary geometry just like a computer game. The second author wrote the programs for all the figures and gave the proofs of all theorems under the direction of the first author. The last author joined the discussions and arranged for publishing.

§ 2. A program for drawing game

We divide our operation into the drawing part and the printing part, due to the circumstances of our computer machines.

PART ONE : To draw a triangle and related figures.
A program for drawing game of Theorem 1 in § 3 is written in Visual Basic Ver. 6.0 by Microsoft Corporation and consists of the following five steps (see **List 1**).

Step 1. Set the coordinates axes and the initial position each vertex of a triangle $\triangle ABC$ and draw each side in black.

Step 2. Plot the orthocenter H in red, the circum-circle with the circumcenter O in blue, and the inscribed circle with the incenter I in green. Draw the straight line through H and O in red.

Step 3. According to the indication by the mouse, move each vertex A, B, C of the triangle from the initial position to an arbitrary position.

Step 4. Draw a perpendicular from each vertex A, B, C to the opposite side with a dotted blue line.

Step 5. When $\triangle ABC$ is obtuse, extend each side up to H with a dotted green line.

PART TWO : To process a bitmap file (bmp) by pLaTeX 2ε, to exhibit it on another display, and to print it.

§ 3. Results

Let V be a point arbitrarily fixed on the plane and $\triangle ABC$ be a triangle on the same plane. Any point P on the plane is uniquely determined by a continued ratio $(\lambda : \mu : \nu)$ where

$$\overrightarrow{VP} = \frac{\lambda \overrightarrow{VA} + \mu \overrightarrow{VB} + \nu \overrightarrow{VC}}{\lambda + \mu + \nu}.$$

The ratio $(\lambda : \mu : \nu)$ is called the barycentric coordinates of P with respect to $\triangle ABC$ (see [**4**; P.164-165], [**1**; P.217-218]).

For example, the barycentric coordinates of a vertex A is given by $(1 : 0 : 0)$. In $\triangle ABC$, put $a = BC$, $b = CA$, $c = AB$ as usual. Then the barycentric coordinates of significant points in $\triangle ABC$ are given as follows :

$(1 : 1 : 1)$	for the center of gravity G;
$(\sin 2A : \sin 2B : \sin 2C)$	for the circumcenter O;
$(a : b : c)$	for the incenter I;
$(-a : b : c)$	for the excenter I_a in the interior of $\angle A$;
$(a : -b : c)$	for the excenter I_b in the interior of $\angle B$;
$(a : b : -c)$	for the excenter I_c in the interior of $\angle C$;
$(\tan A : \tan B : \tan C)$	for the orthocenter H, if $\triangle ABC$ is not a right triangle.

Observing the figures drawn on a display, we obtain the following theorems. In each figure of the theorems the third distinguished point is slightly moved from the straight line through the first and second distinguished points for illustration.

Theorem 1. *If the circumcenter O, the orthocenter H and the incenter I of $\triangle ABC$ are collinear, then $\triangle ABC$ is an isosceles triangle* (**Fig. 1**).

Proof. By the assumption the incenter I lies on the Euler line OGH. We have

$$\overrightarrow{GO} = t\,\overrightarrow{GI}$$

for some scalar t.
Hence

$$\frac{(\sin 2A)\,\overrightarrow{GA}+(\sin 2B)\,\overrightarrow{GB}+(\sin 2C)\,\overrightarrow{GC}}{\sin 2A+\sin 2B+\sin 2C} = t\left(\frac{a\overrightarrow{GA}+b\overrightarrow{GB}+c\overrightarrow{GC}}{a+b+c}\right).$$

We have

$$\frac{(\sin 2A-\sin 2C)\,\overrightarrow{GA}+(\sin 2B-\sin 2C)\,\overrightarrow{GB}}{\sin 2A+\sin 2B+\sin 2C} = t\left\{\frac{(a-c)\overrightarrow{GA}+(b-c)\overrightarrow{GB}}{a+b+c}\right\},$$

since $\overrightarrow{GC}=-\overrightarrow{GA}-\overrightarrow{GB}$.
Hence

$$(\sin 2A-\sin 2C):(\sin 2B-\sin 2C)=(a-c):(b-c),$$

since the vectors $\overrightarrow{GA}$ and $\overrightarrow{GB}$ are linearly independent.
By the law of sines for $\triangle ABC$,

$$(a\cos A-c\cos C):(b\cos B-c\cos C)=(a-c):(b-c).$$

By the law of cosines for $\triangle ABC$,

$$a\left(\frac{b^2+c^2-a^2}{2bc}\right)-c\left(\frac{a^2+b^2-c^2}{2ab}\right):b\left(\frac{c^2+a^2-b^2}{2ca}\right)-c\left(\frac{a^2+b^2-c^2}{2ab}\right)$$
$$=(a-c):(b-c).$$

We have

$$(a-b)(b-c)(c-a)(a+b+c)^2=0.$$

Therefore,

$$a=b \quad \text{or} \quad b=c \quad \text{or} \quad c=a. \quad \square$$

Theorem 2. *If a vertex A, the circumcenter O and the incenter I of $\triangle ABC$ are collinear, then $\triangle ABC$ is an isosceles triangle with $AB = AC$* (**Fig. 2**).

Proof. By the assumption we have

$$\overrightarrow{AO} = t\,\overrightarrow{AI}$$

for some scalar t.
Hence

$$\frac{(\sin 2B)\,\overrightarrow{AB}+(\sin 2C)\,\overrightarrow{AC}}{\sin 2A+\sin 2B+\sin 2C} = t\left(\frac{b\overrightarrow{AB}+c\overrightarrow{AC}}{a+b+c}\right).$$

We have

$$\sin 2B \ :\ \sin 2C = \mathrm{b} : \mathrm{c}\,.$$

By the laws of sines and cosines for $\triangle ABC$,

$$\cos B = \cos C\,.$$

Therefore,

$$\angle B = \angle C\,. \quad \square$$

Theorem 3. ([**1**]) *If a vertex A, the circumcenter O and the center of gravity G of $\triangle ABC$ are collinear, then $\triangle ABC$ is an isosceles triangle with $AB = AC$ or a right triangle with $\angle A = 90^\circ$* (**Figs. 3.1 - 3.2**).

Theorem 3 is given as Exercise 8 of § 1.6 in [**1**]. We give a proof for completeness.

Proof. By the assumption we have

$$\overrightarrow{AO} = t\,\overrightarrow{AG}$$

for some scalar t.
Hence

$$\frac{(\sin 2B)\,\overrightarrow{AB}+(\sin 2C)\,\overrightarrow{AC}}{\sin 2A+\sin 2B+\sin 2C} = t\left(\frac{\overrightarrow{AB}+\overrightarrow{AC}}{3}\right).$$

Hence

$$\sin 2B \ :\ \sin 2C = 1 : 1\,.$$

Therefore,

$$\angle B = \angle C \quad \text{or} \quad \angle A = 90^\circ . \quad \square$$

Theorem 4. *If a vertex A, the circumcenter O and the orthocenter H of $\triangle ABC$ are collinear, then $\triangle ABC$ is an isosceles triangle with AB = AC or a right triangle with* $\angle A = 90^\circ$ (**Fig. 4**).

Proof. *Case 1.* *$\triangle ABC$ is not a right triangle.*
By the assumption we have

$$\overrightarrow{AO} = t\,\overrightarrow{AH}$$

for some scalar t.
Hence

$$\frac{(\sin 2B)\,\overrightarrow{AB} + (\sin 2C)\,\overrightarrow{AC}}{\sin 2A + \sin 2B + \sin 2C} = t\left\{\frac{(\tan B)\,\overrightarrow{AB} + (\tan C)\,\overrightarrow{AC}}{\tan A + \tan B + \tan C}\right\}.$$

Hence

$$\sin 2B \;:\; \sin 2C = \tan B \;:\; \tan C .$$

By the laws of sines and cosines for $\triangle ABC$,

$$\cos B = \cos C .$$

Therefore,

$$\angle B = \angle C .$$

Case 2. $\angle A = 90^\circ$. The points A, O and H are collinear.
Case 3. $\angle B = 90^\circ$ *or* $\angle C = 90^\circ$. The points A, O and H are not collinear. $\square$

Theorem 5. *If the center of gravity G, the incenter I and the excenter I_a in the interior of $\angle A$ in $\triangle ABC$ are collinear, then $\triangle ABC$ is an isosceles triangle with AB = AC* (**Fig. 5**).

Proof. Since the vertex A lies on the straight line IGI_a, we have

$$\overrightarrow{AI} = t\,\overrightarrow{AG}$$

for some scalar t.

Hence

$$\frac{b\,\overrightarrow{AB}+c\,\overrightarrow{AC}}{a+b+c} = t\left(\frac{\overrightarrow{AB}+\overrightarrow{AC}}{3}\right).$$

Hence

$$b : c = 1 : 1.$$

Therefore,

$$b = c. \quad \square$$

Theorem 6. *If the orthocenter H, the incenter I and the excenter I_a in the interior of $\angle A$ in $\triangle ABC$ are collinear, then $\triangle ABC$ is an isosceles triangle with $AB = AC$ or a right triangle with $\angle A = 90^\circ$* (**Fig. 6**).

Proof. *Case 1.* *$\triangle ABC$ is not a right triangle.*
Since the vertex A lies on the straight line IHI_a, we have

$$\overrightarrow{AH} = t\,\overrightarrow{AI}$$

for some scalar t.
Hence

$$\frac{(\tan B)\,\overrightarrow{AB}+(\tan C)\,\overrightarrow{AC}}{\tan A+\tan B+\tan C} = t\left(\frac{b\,\overrightarrow{AB}+c\,\overrightarrow{AC}}{-a+b+c}\right).$$

Hence

$$\tan B : \tan C = b : c.$$

By the laws of sines and cosines for $\triangle ABC$,

$$\cos B = \cos C.$$

Therefore,

$$\angle B = \angle C.$$

Case 2. $\angle A = 90^\circ$. The points H, I and I_a are collinear.
Case 3. $\angle B = 90^\circ$ *or* $\angle C = 90^\circ$. The points H, I and I_a are not collinear. $\square$

Theorem 7. *If the orthocenter H and the excenters I_b, I_c in the interior of* $\angle B$, $\angle C$ *in* $\triangle ABC$, *respectively, are collinear, then* $\triangle ABC$ *is a right triangle with* $\angle A = 90^\circ$ (**Fig. 7**).

Proof. *Case 1.* $\triangle ABC$ *is not a right triangle.*
Since the vertex A lies on the straight line $I_b H I_c$, we have

$$\overrightarrow{AH} = t\,\overrightarrow{AI_b}$$

for some scalar t.

Hence

$$\frac{(\tan B)\,\overrightarrow{AB}+(\tan C)\,\overrightarrow{AC}}{\tan A+\tan B+\tan C} = t\left\{\frac{(-b)\overrightarrow{AB}+c\,\overrightarrow{AC}}{a-b+c}\right\}.$$

Hence

$$\frac{\tan B}{\tan C} = \frac{-b}{c}.$$

In the above equation the left-hand side >0 and the right-hand side <0, which is a contradiction.

Case 2. $\angle A = 90^\circ$. The points H, I_b and I_c are collinear.

Case 3. $\angle B = 90^\circ$ *or* $\angle C = 90^\circ$. The points H, I_b and I_c are not collinear. □

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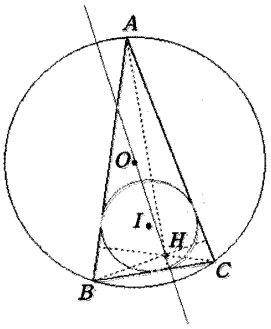

Fig. 1

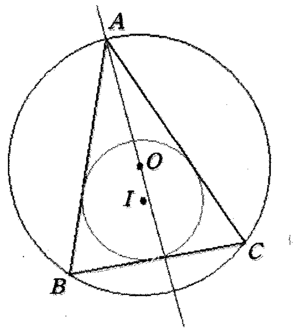

Fig. 2

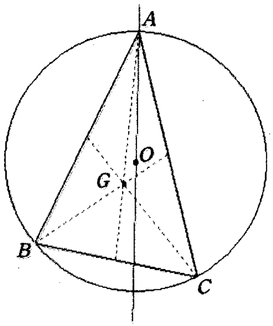

Fig. 3.1

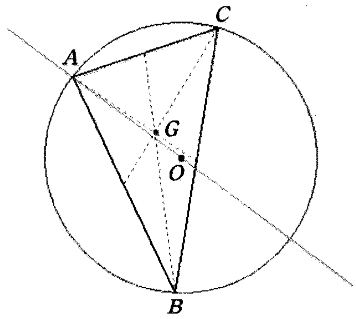

Fig. 3.2

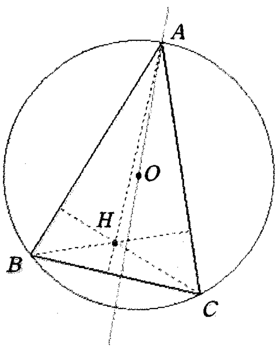

Fig. 4

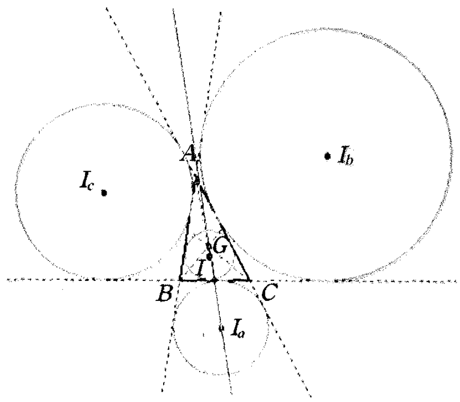

Fig. 5

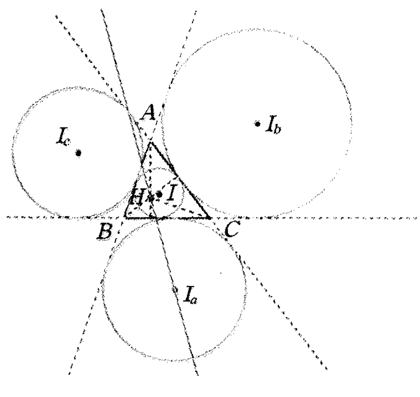

Fig. 6

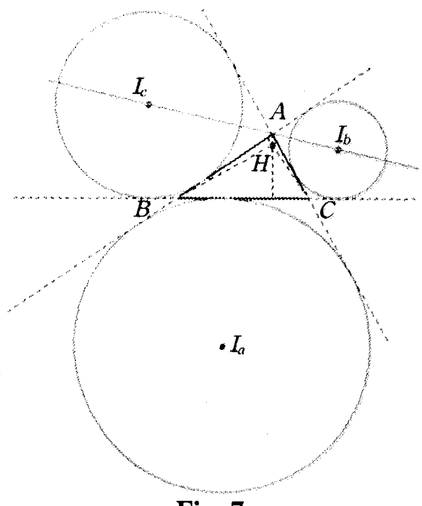

Fig. 7

List 1. A program for drawing game of Theorem 1

```
Dim sx As Double, ex As Double
Dim sy As Double, ey As Double
Dim ax As Double, ay As Double
Dim bx As Double, by As Double
Dim cx As Double, cy As Double
Dim k As Integer
Dim ra(4) As Double
Dim hx, hy, ox, oy As Double
Dim xh, yh, xo, yo As Double
Dim qxh, qyh, qxo, qyo As Double

'------------------------------
'    Drawing the initial figure
'------------------------------
Private Sub Form_Activate()
  Form1.DrawMode = vbNotXorPen
  Form1.Line (ax, ay)-(bx, by)
  Form1.Line (bx, by)-(cx, cy)
  Form1.Line (cx, cy)-(ax, ay)
  Orthocenter ax, ay, bx, by, cx, cy
  Circum ax, ay, bx, by, cx, cy
  Incircle ax, ay, bx, by, cx, cy
  Ed_Line xh, yh, xo, yo, 12
  qxh = xh: qyh = yh
  qxo = xo: qyo = yo
End Sub

'-------------------------------------------------------
'    Setting the form, the coordinate axes and the initial
'    values of the coordinates of each vertex of a triangle
'-------------------------------------------------------
Private Sub Form_Load()
  Form1.Top = 100
  Form1.Left = 100
  Form1.Width = 0.6 * Screen.Width
  Form1.Height = 0.8 * Screen.Height
  sx = -15
  ex = 15
  wx = ex - sx
  wy = wx * Form1.ScaleHeight / Form1.ScaleWidth
  sy = -0.5 * wy
  ey = 0.5 * wy
  Form1.Scale (sx, ey)-(ex, sy)
  Form1.BackColor = vbWhite
  Form1.DrawWidth = 1
  ax = -3: ay = 8
  bx = -5: by = -5
  cx = 6: cy = 2
  k = 1
End Sub
```

```
'---------------------------------------------------
'    Action corresponding to the left button of mouse
'---------------------------------------------------
Private Sub Form_MouseDown(Button As Integer, Shift As Integer, X As Single, Y As Single)
  If k = 2 Then
    k = 1
    Exit Sub
  End If

  k = 2
  d1 = Sqr((ax - X) ^ 2 + (ay - Y) ^ 2)
  If d1 < 1 Then
    Form1.Line (ax, ay)-(bx, by)
    Form1.Line (cx, cy)-(ax, ay)
    Form1.Line (X, Y)-(bx, by)
    Form1.Line (cx, cy)-(X, Y)
    Orthocenter ax, ay, bx, by, cx, cy
    Orthocenter X, Y, bx, by, cx, cy
    Circum ax, ay, bx, by, cx, cy
    Circum X, Y, bx, by, cx, cy
    Incircle ax, ay, bx, by, cx, cy
    Incircle X, Y, bx, by, cx, cy
    Ed_Line qxh, qyh, qxo, qyo, 12
    Ed_Line xh, yh, xo, yo, 12
    qxh = xh: qyh = yh
    qxo = xo: qyo = yo
    ax = X: ay = Y
    Exit Sub
  End If

  d2 = Sqr((bx - X) ^ 2 + (by - Y) ^ 2)
  If d2 < 1 Then
    Form1.Line (bx, by)-(ax, ay)
    Form1.Line (cx, cy)-(bx, by)
    Form1.Line (X, Y)-(ax, ay)
    Form1.Line (cx, cy)-(X, Y)
    Orthocenter ax, ay, bx, by, cx, cy
    Orthocenter ax, ay, X, Y, cx, cy
    Circum ax, ay, bx, by, cx, cy
    Circum ax, ay, X, Y, cx, cy
    Incircle ax, ay, bx, by, cx, cy
    Incircle ax, ay, X, Y, cx, cy
    Ed_Line qxh, qyh, qxo, qyo, 12
    Ed_Line xh, yh, xo, yo, 12
    qxh = xh: qyh = yh
    qxo = xo: qyo = yo
    bx = X: by = Y
    Exit Sub
  End If

  d3 = Sqr((cx - X) ^ 2 + (cy - Y) ^ 2)
  If d3 < 1 Then
    Form1.Line (cx, cy)-(bx, by)
    Form1.Line (ax, ay)-(cx, cy)
```

```
    Form1.Line (X, Y)-(bx, by)
    Form1.Line (ax, ay)-(X, Y)
    Orthocenter ax, ay, bx, by, cx, cy
    Orthocenter ax, ay, bx, by, X, Y
    Circum ax, ay, bx, by, cx, cy
    Circum ax, ay, bx, by, X, Y
    Incircle ax, ay, bx, by, cx, cy
    Incircle ax, ay, bx, by, X, Y
    Ed_Line qxh, qyh, qxo, qyo, 12
    Ed_Line xh, yh, xo, yo, 12
    qxh = xh: qyh = yh
    qxo = xo: qyo = yo
    cx = X: cy = Y
  End If
End Sub

'--------------------------------------------------
'     Action corresponding to the movement of mouse
'--------------------------------------------------
Private Sub Form_MouseMove(Button As Integer, Shift As Integer, X As Single, Y As Single)
  If k = 1 Then Exit Sub

  d1 = Sqr((ax - X) ^ 2 + (ay - Y) ^ 2)
  If d1 < 1 Then
    Form1.Line (ax, ay)-(bx, by)
    Form1.Line (cx, cy)-(ax, ay)
    Form1.Line (X, Y)-(bx, by)
    Form1.Line (cx, cy)-(X, Y)
    Orthocenter ax, ay, bx, by, cx, cy
    Orthocenter X, Y, bx, by, cx, cy
    Circum ax, ay, bx, by, cx, cy
    Circum X, Y, bx, by, cx, cy
    Incircle ax, ay, bx, by, cx, cy
    Incircle X, Y, bx, by, cx, cy
    Ed_Line qxh, qyh, qxo, qyo, 12
    Ed_Line xh, yh, xo, yo, 12
    qxh = xh: qyh = yh
    qxo = xo: qyo = yo
    ax = X: ay = Y
    Exit Sub
  End If

  d2 = Sqr((bx - X) ^ 2 + (by - Y) ^ 2)
  If d2 < 1 Then
    Form1.Line (bx, by)-(ax, ay)
    Form1.Line (cx, cy)-(bx, by)
    Form1.Line (X, Y)-(ax, ay)
    Form1.Line (cx, cy)-(X, Y)
    Orthocenter ax, ay, bx, by, cx, cy
    Orthocenter ax, ay, X, Y, cx, cy
    Circum ax, ay, bx, by, cx, cy
    Circum ax, ay, X, Y, cx, cy
    Incircle ax, ay, bx, by, cx, cy
```

```
      Incircle ax, ay, X, Y, cx, cy
      Ed_Line qxh, qyh, qxo, qyo, 12
      Ed_Line xh, yh, xo, yo, 12
      qxh = xh: qyh = yh
      qxo = xo: qyo = yo
      bx = X: by = Y
      Exit Sub
   End If

   d3 = Sqr((cx - X) ^ 2 + (cy - Y) ^ 2)
   If d3 < 1 Then
      Form1.Line (cx, cy)-(bx, by)
      Form1.Line (ax, ay)-(cx, cy)
      Form1.Line (X, Y)-(bx, by)
      Form1.Line (ax, ay)-(X, Y)
      Orthocenter ax, ay, bx, by, cx, cy
      Orthocenter ax, ay, bx, by, X, Y
      Circum ax, ay, bx, by, cx, cy
      Circum ax, ay, bx, by, X, Y
      Incircle ax, ay, bx, by, cx, cy
      Incircle ax, ay, bx, by, X, Y
      Ed_Line qxh, qyh, qxo, qyo, 12
      Ed_Line xh, yh, xo, yo, 12
      qxh = xh: qyh = yh
      qxo = xo: qyo = yo
      cx = X: cy = Y
   End If
End Sub

'---------------------------------------------
'     Presentation of the orthocenter of a triangle
'---------------------------------------------
Private Sub Orthocenter(jxa, jya, jxb, jyb, jxc, jyc)
   xa = jxa: ya = jya
   xb = jxb: yb = jyb
   xc = jxc: yc = jyc
   ux = xb - xa: uy = yb - ya
   vx = xa - xc: vy = ya - yc
   wx = xc - xb: wy = yc - yb
   a = Sqr(wx ^ 2 + wy ^ 2)
   b = Sqr(vx ^ 2 + vy ^ 2)
   c = Sqr(ux ^ 2 + uy ^ 2)
   ah = b ^ 2 + c ^ 2 - a ^ 2
   bh = c ^ 2 + a ^ 2 - b ^ 2
   ch = a ^ 2 + b ^ 2 - c ^ 2
   If ah >= 0 And bh >= 0 And ch >= 0 Then
      p1 = xb * yc - xc * yb: q1 = -wy * ya - wx * xa
      p2 = xc * ya - xa * yc: q2 = -vy * yb - vx * xb
      p3 = xa * yb - xb * ya: q3 = -uy * yc - ux * xc
      xd = (p1 * wy - q1 * wx) / a ^ 2
      yd = (-p1 * wx - q1 * wy) / a ^ 2
      xe = (p2 * vy - q2 * vx) / b ^ 2
      ye = (-p2 * vx - q2 * vy) / b ^ 2
      xf = (p3 * uy - q3 * ux) / c ^ 2
```

```
    yf = (-p3 * ux - q3 * uy) / c ^ 2
    xh = (q2 * wy - q1 * vy) / (wx * vy - wy * vx)
    yh = (q1 * vx - q2 * wx) / (wx * vy - wy * vx)
    Form1.DrawStyle = 2
    Form1.Line (xa, ya)-(xd, yd), vbBlue
    Form1.Line (xb, yb)-(xe, ye), vbBlue
    Form1.Line (xc, yc)-(xf, yf), vbBlue
    Form1.DrawStyle = 0
    Form1.DrawWidth = 5
    Form1.PSet (xh, yh), QBColor(12)
    Form1.DrawWidth = 1
  Else
    If bh < 0 Then
      dm = xa: xa = xb: xb = dm
      dm = ya: ya = yb: yb = dm
    End If

    If ch < 0 Then
      dm = xa: xa = xc: xc = dm
      dm = ya: ya = yc: yc = dm
    End If

    ux = xb - xa: uy = yb - ya
    vx = xa - xc: vy = ya - yc
    wx = xc - xb: wy = yc - yb
    d = wx * vy - wy * vx
    If Abs(d) > 0.001 Then
      a = Sqr(wx ^ 2 + wy ^ 2)
      b = Sqr(vx ^ 2 + vy ^ 2)
      c = Sqr(ux ^ 2 + uy ^ 2)
      p1 = xb * yc - xc * yb: q1 = -wy * ya - wx * xa
      q2 = -vy * yb - vx * xb
      xd = (p1 * wy - q1 * wx) / a ^ 2
      yd = (-p1 * wx - q1 * wy) / a ^ 2
      xh = (q2 * wy - q1 * vy) / d
      yh = (q1 * vx - q2 * wx) / d
      H_Prolong xb, yb, xa, ya
      H_Prolong xc, yc, xa, ya
      Form1.DrawStyle = 2
      Form1.Line (xh, yh)-(xd, yd), vbBlue
      Form1.Line (xb, yb)-(xh, yh), vbBlue
      Form1.Line (xc, yc)-(xh, yh), vbBlue
      Form1.DrawStyle = 0
      Form1.DrawWidth = 5
      Form1.PSet (xh, yh), QBColor(12)
      Form1.DrawWidth = 1
    End If
  End If
End Sub
```

```
'-------------------------------------------------------------------
'    Extension of two sides at the adjacent vertex with an obtuse angle
'-------------------------------------------------------------------
Private Sub H_Prolong(x1, y1, x2, y2)
  Form1.DrawWidth = 1
  l = Sqr((x1 - x2) ^ 2 + (y1 - y2) ^ 2)

  ra(1) = Sqr((x1 - sx) ^ 2 + (y1 - sy) ^ 2)
  ra(2) = Sqr((x1 - ex) ^ 2 + (y1 - sy) ^ 2)
  ra(3) = Sqr((x1 - ex) ^ 2 + (y1 - ey) ^ 2)
  ra(4) = Sqr((x1 - sx) ^ 2 + (y1 - ey) ^ 2)

  r = ra(1)
  For i = 2 To 4
    If ra(i) > r Then r = ra(i)
  Next i

  p = r / l
  tx = (1 - p) * x1 + p * x2
  ty = (1 - p) * y1 + p * y2
  Form1.DrawStyle = 2
  Form1.Line (x2, y2)-(tx, ty), QBColor(2)
  Form1.DrawStyle = 0
End Sub

'--------------------------------------------
'    Presentation of the circumcenter of a triangle
'--------------------------------------------
Private Sub Circum(xa, ya, xb, yb, xc, yc)
  mx = (xa + xb) / 2: my = (ya + yb) / 2
  nx = (xa + xc) / 2: ny = (ya + yc) / 2
  ux = xb - xa: uy = yb - ya
  vx = xc - xa: vy = yc - ya
  wx = xc - xb: wy = yc - yb
  d = ux * vy - vx * uy
  If Abs(d) > 0.01 Then
    a = Sqr(wx ^ 2 + wy ^ 2)
    b = Sqr(vx ^ 2 + vy ^ 2)
    c = Sqr(ux ^ 2 + uy ^ 2)
    s = (a + b + c) / 2
    Ls = Sqr(s * (s - a) * (s - b) * (s - c))
    r = (a * b * c) / (4 * Ls)
    p = ux * mx + uy * my
    q = vx * nx + vy * ny
    xo = (p * vy - q * uy) / d
    yo = (q * ux - p * vx) / d
    Form1.DrawWidth = 5
    Form1.PSet (xo, yo), vbBlue
    Form1.DrawWidth = 1
    Form1.Circle (xo, yo), r, vbBlue
  End If
End Sub
```

```
'-----------------------------------------
'      Presentation of the incenter of a triangle
'-----------------------------------------
Private Sub Incircle(xa, ya, xb, yb, xc, yc)
  ux = xb - xa: uy = yb - ya
  vx = xc - xa: vy = yc - ya
  wx = xc - xb: wy = yc - yb
  d = ux * vy - vx * uy
  If Abs(d) > 0.01 Then
    a = Sqr(wx ^ 2 + wy ^ 2)
    b = Sqr(vx ^ 2 + vy ^ 2)
    c = Sqr(ux ^ 2 + uy ^ 2)
    s = (a + b + c) / 2
    Ls = Sqr(s * (s - a) * (s - b) * (s - c))
    r = Ls / s
    p1 = ya * xb - yb * xa
    Lf = uy * xc - ux * yc + p1
    e = -Lf / Abs(Lf)
    p2 = ya * xc - yc * xa
    Lg = vy * xb - vx * yb + p2
    f = -Lg / Abs(Lg)
    q1 = p1 + e * r * c
    q2 = p2 + f * r * b
    Xi = (q1 * vx - q2 * ux) / (ux * vy - uy * vx)
    Yi = (q1 * vy - q2 * uy) / (ux * vy - uy * vx)
    Form1.DrawWidth = 5
    Form1.PSet (Xi, Yi), QBColor(2)
    Form1.DrawWidth = 1
    Form1.Circle (Xi, Yi), r, vbBlue
  End If
End Sub

'-----------------------------------------------------
'      Drawing the straight line passing through two points
'-----------------------------------------------------
Private Sub Ed_Line(x1, y1, x2, y2, c)
  If Abs(x1 - x2) <= Abs(y1 - y2) Then
    edx1 = ((x1 - x2) * sy - x1 * y2 + x2 * y1) / (y1 - y2)
    edy1 = sy
    edx2 = ((x1 - x2) * ey - x1 * y2 + x2 * y1) / (y1 - y2)
    edy2 = ey
  Else
    edx1 = sx
    edy1 = ((y1 - y2) * sx + x1 * y2 - x2 * y1) / (x1 - x2)
    edx2 = ex
    edy2 = ((y1 - y2) * ex + x1 * y2 - x2 * y1) / (x1 - x2)
  End If

  Form1.Line (edx1, edy1)-(edx2, edy2), QBColor(c)
End Sub
```