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A Triangle with Distinguished Concyclic Points

— Instruction of Geometry by Use of a Drawing Game on a Display —

by

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Abstract

An effective use of a computer makes mathematics classes much more interesting and motivates many students to learn.

In elementary geometry we have five significant notions for a triangle; that is, the center of gravity, the incenter, the excenter, the circumcenter and the orthocenter. Concerning those five points we have a natural problem. Find a property of a triangle on the plane such that some of those five points and two vertices lie on a common circle.

In this paper we present a drawing game for a student to find a solution to the above problem with fun, and obtain some results from our various experiments.

§ 1. Introduction

In the sequel to [4], [5] our study aims to develop a drawing game on a display as a teaching material for elementary geometry classes with activities using computers.

In elementary geometry we have five significant notions for a triangle; that is, the center of gravity, the incenter, the excenter, the circumcenter and the orthocenter. Concerning those points we have a natural problem. Find a property of a triangle on the plane such that some of those five points and two vertices lie on a common circle.

In this paper we present a drawing game for a student to find a solution to the above problem with fun, and obtain some results from our various experiments. In fact, "Mathematical Discovery" due to G. Polya [**9**] indicates the spirit of our research in mathematics education.

For terminology of geometry throughout the paper, consult [**2**], [**3**] and [**8**].

As for the present paper, the first author gave a general idea for drawing a triangle and related figures on a display and made some preliminary programs for teaching materials in elementary geometry just like a computer game. The second author wrote the programs for all the figures and gave the proofs of all theorems under the direction of the first author. The last author joined the discussions and arranged for publishing.

§ 2. A program for drawing game

We divide our operation into the drawing part and the printing part, due to the circumstances of our computer machines.

PART ONE : To draw a triangle and related figures.
A program for drawing game which describes the properties obtained in Theorem 1 in § 3 is written in Visual Basic Ver. 6.0 by Microsoft Corporation and consists of the following seven steps (see **List 1**).

Step 1. Set the coordinates axes and the initial position of each vertex of a triangle $\triangle ABC$ and draw each side in black.

Step 2. Plot the orthocenter H in red, the circum-circle with the circumcenter O in blue, and the inscribed circle with the incenter I in green.

Step 3. Plot three excircles in ocher and three excenters I_a, I_b, I_c, in red.

Step 4. Plot the circle determined by I, I_b and I_c in pink.

Step 5. According to the indication by the mouse, move each vertex A, B, C of the triangle from the initial position to an arbitrary position.

Step 6. Draw a perpendicular from each vertex A, B, C to the opposite side with a dotted blue line.

Step 7. When $\triangle ABC$ is obtuse, extend each side up to H with a dotted green line.

PART TWO : To process a bitmap file (bmp) by pLaTeX 2ε, to exhibit it on another display, and to print it.

§ 3. Results

Theorem 1. *In $\triangle ABC$ let S be the center of the circle determined by the incenter I and the excenters I_b, I_c, in the interiors of $\angle B$, $\angle C$, respectively. Then we have the following :*

(1) (*The radius of the circle* S) = 2× (*the radius of the circumcircle of* $\triangle ABC$),

(2) *The excenter I_a in the interior of $\angle A$, the circumcenter O and the point S are collinear,*

(3) $I_aO = OS$ (**Fig. 1**).

Theorem 2. *Let S be the center of the circle determined by the vertices B, C and the orthocenter H of $\triangle ABC$. If the circle S passes through the circumcenter O and the incenter I, then we have the following :*

(1) $HI = IO$,

(2) *The circle S passes through the point I_a,*

(3) $HI_a = I_aO$,

where I_a denotes the the excenter in the interior of $\angle A$ (**Fig. 2**).

Theorem 3. *Let S be the center of the circle determined by the vertices B, C and the circumcenter O of $\triangle ABC$. If the circle S passes through the incenter I, then we have the following :*

(1) *The circle S passes through the orthocenter H,*

(2) $HI = IO$,

(3) $HI_a = I_aO$, *where I_a denotes the excenter in the interior of $\angle A$* (**Fig. 3**).

Theorem 4. *Let S be the center of the circle determined by the vertices B, C and the orthocenter H of $\triangle ABC$. If the circle S passes through the incenter I, then we have the following :*

(1) *The circle S passes through the circumcenter O,*

(2) $HI = IO$,

(3) $HI_a = I_aO$, *where I_a denotes the excenter in the interior of $\angle A$* (**Fig. 4**).

Theorem 5. *Let S be the center of the circle determined by the vertices B, C of $\triangle ABC$ and the excenter I_a in the interior of $\angle A$. If the circle S passes through the circumcenter O, then we have the following :*

(1) *The circle passes through the orthocenter H,*

(2) $HI = IO$,

(3) $HI_a = I_aO$ (**Fig. 5**).

Theorem 6. *Let S be the center of the circle determined by the vertices B, C and the orthocenter H of $\triangle ABC$. If the circle S passes the excenter I_a in the interior of $\angle A$, then we have the following :*

(1) *The circle S passes through the incenter I,*

(2) *The circle S passes through the circumcenter O,*

(3) $HI = IO$,

(4) $HI_a = I_aO$ (**Fig. 6**).

Theorem 7. *Let S be the center of the circle determined by the vertices B, C and the orthocenter H of $\triangle ABC$. If the circle S passes through the excenter I_b in the interior of $\angle B$, then we have the following :*

(1) *The circle S passes through the circumcenter O,*

(2) *The circle S passes through the the excenter I_c in the interior of $\angle C$,*

(3) *The line-segment I_bI_c is a diameter of the circle S,*

(4) $HI_b = I_bO$,

(5) $HI_c = I_cO$,

(6) *The lines* OH *and* I_bI_c *are perpendicular to each other* (**Fig. 7**).

Theorem 8. *Let* S *be the center of the circle determined by the vertices* B, C *and the circumcenter* O *of* $\triangle ABC$. *If the circle* S *passes through the excenter* I_b *in the interior of* $\angle B$, *then we have the following :*

(1) *The circle* S *passes through the orthocenter* H,

(2) *The circle* S *passes through the the excenter* I_c *in the interior of* $\angle C$ (**Fig. 8**).

§ 4. Proofs

From now on we work on the complex plane. The unit circle means the circle with center O and radius 1, which is given by the equation $|z| = 1$. Denote the point A represented by a complex number α by $A(\alpha)$.

Throughout this section, for proving the Theorems in § 3, we assume that the vertices $A(\alpha)$, $B(\beta)$, $C(\gamma)$ of a triangle $\triangle ABC$ lie on the unit circle. Put $a = BC$, $b = CA$, $c = AB$ as usual. Then each of the significant points in $\triangle ABC$ is represented by a complex number as follows :

$g = \dfrac{\alpha + \beta + \gamma}{3}$ for the center of gravity G;

0 for the circumcenter O;

$\mu = \dfrac{a\alpha + b\beta + c\gamma}{a + b + c}$ for the incenter I;

$\mu_a = \dfrac{(-a)\alpha + b\beta + c\gamma}{(-a) + b + c}$ for the excenter I_a in the interior of $\angle A$;

$\mu_b = \dfrac{a\alpha + (-b)\beta + c\gamma}{a + (-b) + c}$ for the excenter I_b in the interior of $\angle B$;

$\mu_c = \dfrac{a\alpha + b\beta + (-c)\gamma}{a + b + (-c)}$ for the excenter I_c in the interior of $\angle C$;

$h = \dfrac{\alpha \tan A + \beta \tan B + \gamma \tan C}{\tan A + \tan B + \tan C}$ for the orthocenter H, if $\triangle ABC$ is not a right triangle.

Lemma 1. *For $\triangle ABC$, we have the following :*

(1) ([**7**; P.169, Example 1], [**10**; P.15, Exercise]) *For the orthocenter $H(h)$,*

$$h = \alpha + \beta + \gamma .$$

(2) *The circle determined by the points B, C, H is given by the equation*

$$|z - \beta - \gamma| = 1 .$$

Proof. Let S be the reflection of the point O across the line BC.
(1) Since the quadrangle $OBSC$ is a rhombus, the point S is represented by the complex number $\beta + \gamma$.
Since the quadrangle $OAHS$ is a parallelogram, the point H is represented by the complex number $h = \alpha + \beta + \gamma$.
(2) The equation $|z - \beta - \gamma| = 1$ holds for $z = \beta$, γ, or $\alpha + \beta + \gamma$. □

Lemma 2. *For $\triangle ABC$, we have the following :*

(1) $\alpha\overline{\beta} + \overline{\alpha}\beta = 2 - c^2$,

$\beta\overline{\gamma} + \overline{\beta}\gamma = 2 - a^2$,

$\gamma\overline{\alpha} + \overline{\gamma}\alpha = 2 - b^2$.

(2) $|a\alpha + b\beta + c\gamma|^2 = (a+b+c)(a+b+c-abc)$,

$|a\alpha - b\beta + c\gamma|^2 = (a-b+c)(a-b+c+abc)$,

$|a\alpha + b\beta - c\gamma|^2 = (a+b-c)(a+b-c+abc)$.

(3) $|l\alpha + m\beta + n\gamma|^2 = (l+m+n)^2 - a^2mn - b^2nl - c^2lm$

for real numbers l, m, n.

An easy proof is omitted.

Proof of Theorem 1. Without loss of generality we may add the assumption that the side BC of $\triangle ABC$ is parallel to the imaginary-axis. Setting

$$\begin{cases} \alpha = p + qi \\ \beta = u - vi \\ \gamma = u + vi \end{cases} \quad \text{for real numbers } p,\ q,\ u,\ v \text{ with } v > 0,$$

we have the following :

$$a = |\beta-\gamma| = 2v,$$

$$b = |\gamma-\alpha| = \sqrt{(p-u)^2+(q-v)^2},$$

$$c = |\alpha-\beta| = \sqrt{(p-u)^2+(q+v)^2}.$$

Let S be the reflection of the point I_a across the circumcenter O. To see (1) it suffices to show that

$$SI = SI_b = SI_c = 2.$$

i) $SI = 2$; i.e., $|\mu-(-\mu_a)| = 2$.

By Lemma 1. (1),

$$\mu-(-\mu_a)$$

$$= \frac{a\alpha+b\beta+c\gamma}{a+b+c} + \frac{-a\alpha+b\beta+c\gamma}{-a+b+c}$$

$$= \frac{(a\alpha+b\beta+c\gamma)(-a+b+c)+(-a\alpha+b\beta+c\gamma)(a+b+c)}{(a+b+c)(-a+b+c)}$$

$$= \frac{(-2a^2p+2b^2u+4bcu+2c^2u)+(-2a^2q-2b^2v+2c^2v)i}{(b+c)^2-a^2}.$$

Let $r = |\mu-(-\mu_a)|$. Find the Groebner bases by applying Mathematica by Wolfram Research, Inc. to the following code (see [**1**], [**6**]):

```
f1:=p^2+q^2-1
f2:=u^2+v^2-1
f3:=a-2*v
f4:=b^2-(p-u)^2-(q-v)^2
f5:=c^2-(p-u)^2-(q+v)^2
f6:=((b+c)^2-a^2)*x-(-2*a^2*p+2*b^2*u+4*b*c*u+2*c^2*u)
f7:=((b+c)^2-a^2)*y-(-2*a^2*q-2*b^2*v+2*c^2*v)
f8:=r^2-x^2-y^2
GroebnerBasis[{f1,f2,f3,f4,f5,f6,f7,f8},{p,q,u,v,a,b,c,y,x,r}]
Factor[%]
```

Then we obtain a term

$$b^2c^2(b+c)^2(-2+r)(2+r)$$

in the list of the generating Groebner bases.
Therefore, we have $r-2=0$, or $r=2$.

Similarly we can prove
ii) $SI_b = 2$, and iii) $SI_c = 2$.

(2) The collinearity of the points I_a, O, S is clear in the above discussion.

(3) We can prove by the similar method. □

Proof of Theorem 2. Without loss of generality we may add the assumption that the side BC of $\triangle ABC$ is parallel to the imaginary-axis and

$$(\text{the real part of } \beta) > 0.$$

Since the circle S passes through the circumcenter O, the equation $|z-\beta-\gamma| = 1$ holds for $z=0$.
Hence $|\beta+\gamma| = 1$.
By Lemmas 1 and 2, we have the following :

$$a = \sqrt{3},$$

$$\beta = \frac{1}{2} - \frac{\sqrt{3}}{2}i,$$

$$\gamma = \frac{1}{2} + \frac{\sqrt{3}}{2}i,$$

$$h = \alpha + \beta + \gamma.$$

Because of $\beta+\gamma = 1$, the equation of the circle S is also given by

$$|z-1| = 1.$$

Since the circle S passes through the incenter I, we have

$$|\mu-1| = 1, \qquad \text{or} \qquad |\mu|^2 = \mu + \overline{\mu}.$$

By Lemma 2, $\alpha + \overline{\alpha} = 1 - bc.$

To see (1) $|h-\mu| = \mu$ it suffices to show that

$$|h|^2 = h\overline{\mu} + \overline{h}\mu .$$

We can easily see that both sides equal to $3 - bc.$

To see (2) $|\mu_a - 1| = 1$ it suffices to show that

$$|\mu_a|^2 = \mu_a + \overline{\mu}_a .$$

By Lemma 2, we can prove that both sides equal to $\dfrac{-a+b+c+abc}{-a+b+c}$.

(3) A similar proof is omitted. □

Proof of Theorem 3. To see (1) we shall show that the circle S is given by the equation

$$|z-\beta-\gamma| = 1.$$

Let s be a complex number which represents the point S. The circle S is given by $|z-s| = s$, because it passes through the circumcenter O.
Setting $s = t(\beta + \gamma)$ with a real number t, we show that

$$t = 1 \qquad \text{and} \qquad |s| = 1.$$

Since the circle S passes through the vertex B and the incenter I,

$$\begin{cases} |\beta - s| = |s| \\ |\mu - s| = |s|. \end{cases}$$

By Lemma 2. (3),

$$\begin{cases} t\left(4-a^2\right)-1 = 0 \\ (1-4t)(a+b+c)+ta\left\{a(b+c)+b^2+c^2\right\}-abc = 0, \end{cases}$$

or

$$\begin{cases} t\left(4-a^2\right)-1 = 0 \\ t\left(b^2+c^2-a^2\right)-bc = 0. \end{cases}$$

By the law of sines and the law of cosines for $\triangle ABC$,

$$\begin{cases} 4t\cos^2 A - 1 = 0 \\ 2t\cos A - 1 = 0. \end{cases}$$

Since $t \neq 0$, we have the following :

$$t = 1,$$

$$\angle A = 60^\circ,$$

$$a = \sqrt{3},$$

$$|s| = 1.$$

Both (2) and (3) follow from the Theorem 2. □

Proof of Theorem 7. To see (1) we shall show that the equation of the circle S : $|z-\beta-\gamma| = 1$ holds for $z = 0$; i.e., $|\beta+\gamma| = 1$.
Since the circle S passes through the excenter I_b, we have $|\mu_b - \beta - \gamma| = 1$.
By Lemma 2. (3), $a^2 = b^2 + bc + c^2$.
By the law of cosines for $\triangle ABC$,

$$\cos A = -\frac{1}{2}, \qquad \text{or} \qquad A = 120^\circ.$$

Hence $$a = \sqrt{3},$$

Therefore $$|\beta+\gamma| = 1.$$

(2) We show that

$$|\mu_c - \beta - \gamma| = 1,$$

or equivalently

$$\left|(a\alpha + b\beta - c\gamma) - (a+b-c)(\beta+\gamma)\right|^2 = (a+b-c)^2.$$

In fact,

$$\begin{aligned} \text{the left side} &= (a+b-c)^2 + a(a+b-c)\left\{(b^2+bc+c^2) - a^2\right\} && \text{by Lemma 2. (3)}, \\ &= (a+b-c)^2 && \text{by } a^2 = b^2+bc+c^2, \\ &= \text{the right side.} \end{aligned}$$

(3) It suffices to show that

$$|\mu_b - \mu_c| = 2,$$

or equivalently

$$|a(b-c)\alpha - ab\beta + ac\gamma|^2 = (a-b+c)^2(a+b-c)^2.$$

In fact,

$$\begin{aligned}\text{the left side} &= a^2|(b-c)\alpha - b\beta + c\gamma|^2 && \text{by Lemma 2. (3),}\\ &= a^2bc\left\{a^2-(b-c)^2\right\} && \text{by } a^2=3,\\ &= 3bc\left\{a^2-(b-c)^2\right\} && \text{by } a^2=b^2+bc+c^2,\\ &= \left\{a^2-(b-c)^2\right\}^2 \\ &= \text{the right side.}\end{aligned}$$

(4) We show that

$$|h-\mu_b| = |\mu_b|$$

or, equivalently

$$|h-\mu_b|^2 = |\mu_b|^2.$$

By Lemma 2, (3) and $a^2 = b^2 + bc + c^2$, both sides equal to

$$(a-b+c)(a-b+c+abc).$$

(5) A similar proof is omitted.
(6) We show that the inner-product

$$\overrightarrow{OH} \cdot \overrightarrow{I_bI_c} = 0$$

or, equivalently

$$h(\overline{\mu}_c - \overline{\mu}_b) + \overline{h}(\mu_c - \mu_b) = 0.$$

This follows form (4) and (5). □

The proofs for Theorems 4 - 6 and 8 are similar to those of Theorem 3 and 7.

References

[1] W. W. Adams and P. Loustaunau : *An Introduction to Gröbner Bases*, Graduation Studies in Math., Vol. **3**, Amer. Math. Soc., Providence, 1994.

[2] H. S. M. Coxeter : *Introduction to Geometry*, 2nd ed., John Wiley and Sons Inc, New York, 1980.〔コクセター（銀林　浩・訳）: 幾何学入門　第2版，明治図書，東京，1982.〕

[3] H. S. M. Coxeter and S. L. Greitzer : *Geometry revised*, New Mathematical Library, Number **6**, School Mathematics Study Group, Random House, Inc., New York, 1967.〔コークスター，グレイツァー（寺阪英孝・訳）: 幾何学再入門，SMSG 双書，河出書房新社，東京，1970.〕

[4] K. Fujita, A. Matsushima and H. Fukaishi : A Locus of the Orthocenter of a Triangle －Instruction in Geometry by a Moving Locus on a Computer, *Mem. Fac. Educ., Kagawa University II*, **55**(2005), 1-13.

[5] K. Fujita, A. Matsushima and H. Fukaishi : A Triangle with Three Distinguished Collinear Points － Instruction of Geometry by Use of a Drawing Game on a Display, *Mem. Fac. Educ., Kagawa University II*, **55**(2005), 25-41.

[6] 一松　信 : 代数学入門　第三課，近代科学社，東京，1994.〔= Sin Hitotumatu : *Introduction to Algebra, The Third Lesson*, Kindai-kagaku-sha, Tokyo, 1994.〕

[7] 片山孝次 : 複素数の幾何学，岩波書店，東京，1982.〔= K. Katayama : *Geometry of the Complex Numbers,* Iwanami-shoten, Tokyo, 1982.〕

[8] D. Pedoe : *Geometry*, Dover Publ. Inc., New York, 1970.

[9] G. Polya : *Mathematical Discovery on understanding, learning, and teaching problem solving*, Vol. 1 - 2, John Wiley and Sons, Inc., New York, 1962.〔ポリア（柴垣和三雄，金山靖夫・訳）: 数学の問題の発見的解き方，Ⅰ-Ⅱ，みすず書房，東京，1964.〕

[10] 梅沢敏夫，後藤達生 : 複素数と幾何学，培風館，東京，1993.〔= T. Umezawa and T. Goto : *Complex Numbers and Geometry*, Baifukan, Tokyo, 1993.〕

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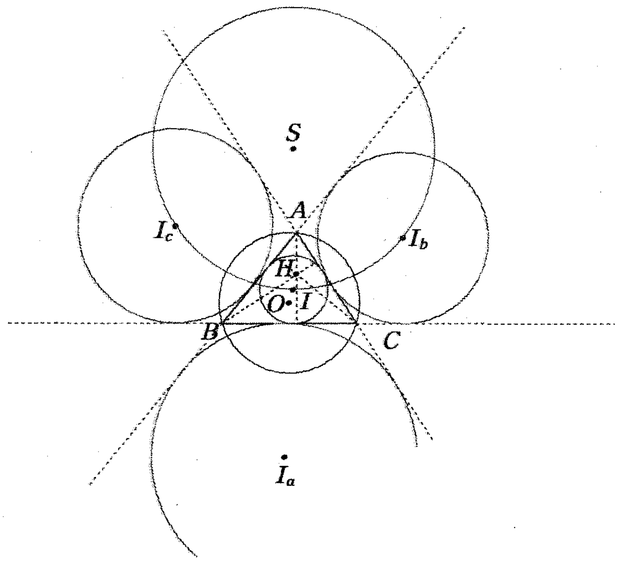


Fig. 1

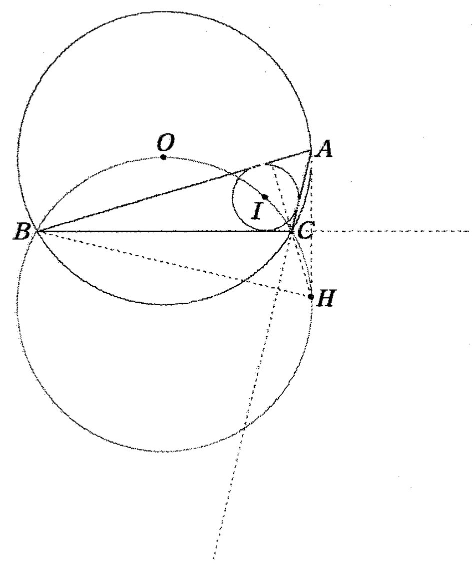
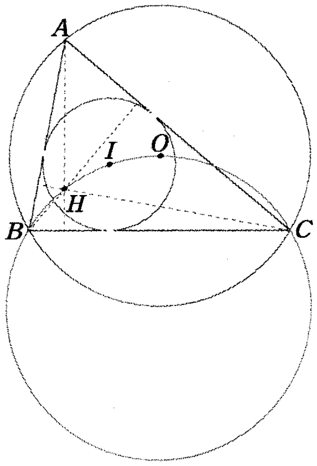


Fig. 2



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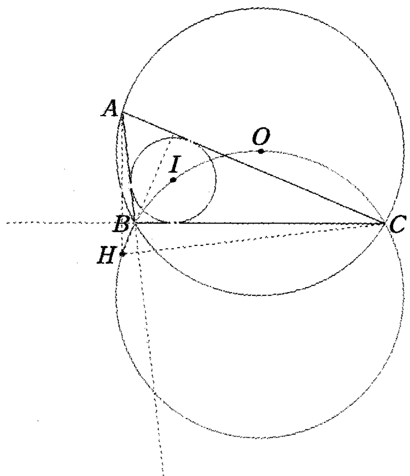

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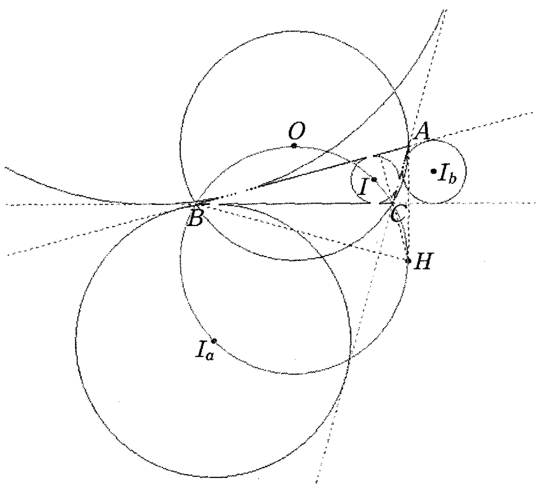

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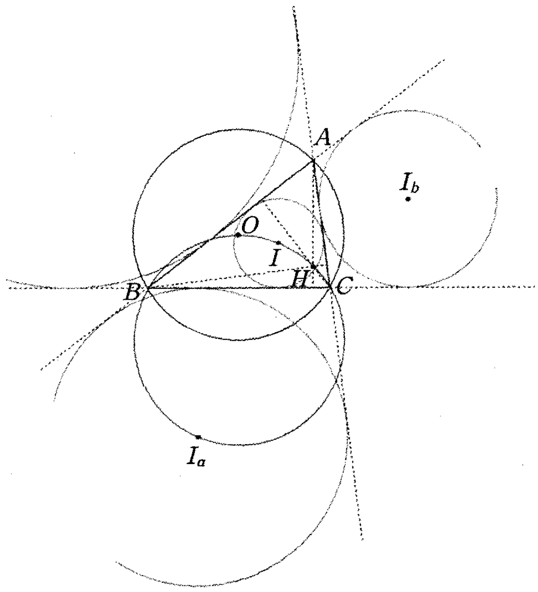

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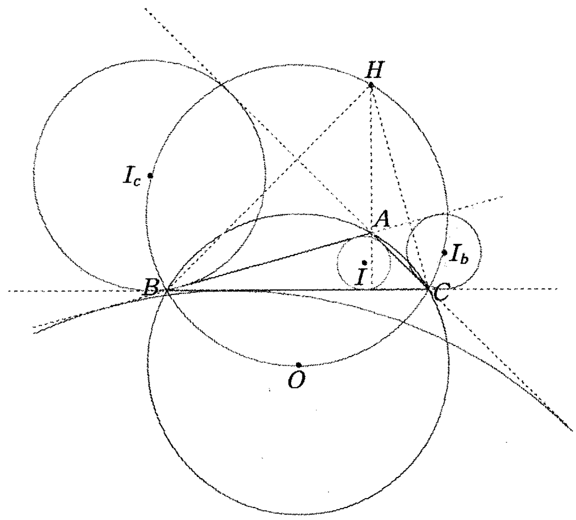

Fig. 7

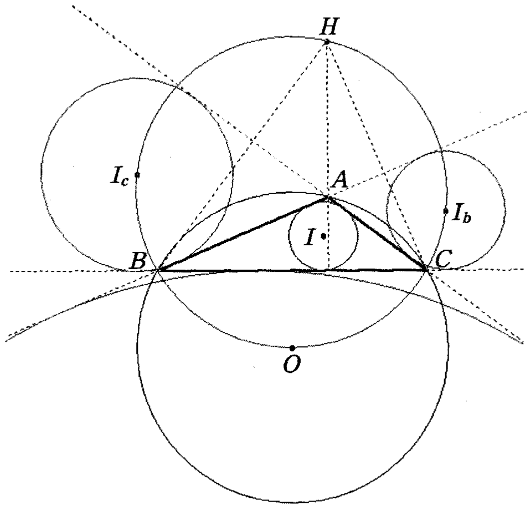

Fig. 8

List 1. A program for drawing game of Theorem 1

```
'----------------------------------------------
'    Specifying statements of the general variables
'----------------------------------------------
Dim sx As Double, ex As Double
Dim sy As Double, ey As Double
Dim ax As Double, ay As Double
Dim bx As Double, by As Double
Dim cx As Double, cy As Double
Dim xia, yia As Double
Dim xi, yi, ebx, eby, ecx, ecy As Double
Dim qxi, qyi, qebx, qeby, qecx, qecy As Double
Dim ra(4) As Double
Dim k As Integer

'-------------------------------
'    Drawing the initial figure
'-------------------------------
Private Sub Form_Activate()
  Form1.DrawMode = vbNotXorPen
  Form1.Line (ax, ay)-(bx, by)
  Form1.Line (bx, by)-(cx, cy)
  Form1.Line (cx, cy)-(ax, ay)
  Prolongs ax, ay, bx, by, cx, cy
  Escribeds ax, ay, bx, by, cx, cy
  Incircle ax, ay, bx, by, cx, cy
  Orthocenter ax, ay, bx, by, cx, cy
  Circum ax, ay, bx, by, cx, cy
  Circ xi, yi, ebx, eby, ecx, ecy, 13
  qxi = xi: qyi = yi
  qebx = ebx: qeby = eby
  qecx = ecx: qecy = ecy
End Sub

'-----------------------------------------------------
'    Setting the form, the coordinate axes and the initial
'    values of the coordinates of each vertex of a triangle
'-----------------------------------------------------
Private Sub Form_Load()
  Top = 500
  Left = 500
  Form1.Width = 0.95 * Screen.Width
  Form1.Height = 0.9 * Screen.Height
  sx = -20
  ex = 20
  wx = ex - sx
  wy = wx * Form1.ScaleHeight / Form1.ScaleWidth
  sy = -0.5 * wy
  ey = 0.5 * wy
  Form1.Scale (sx, ey)-(ex, sy)
  Form1.BackColor = vbWhite
  Form1.DrawWidth = 1
```

```
  ax = -3: ay = 5
  bx = -2: by = -3
  cx = 2: cy = 3
  k = 1
End Sub

'----------------------------------------------------
'      Action corresponding to the left button of mouse
'----------------------------------------------------
Private Sub Form_MouseDown(Button As Integer, Shift As Integer, X As Single, Y As Single)
  If k = 2 Then
    k = 1
    Exit Sub
  End If

  k = 2
  d1 = Sqr((ax - X) ^ 2 + (ay - Y) ^ 2)
  If d1 < 1 Then
    Form1.Line (ax, ay)-(bx, by)
    Form1.Line (cx, cy)-(ax, ay)
    Form1.Line (X, Y)-(bx, by)
    Form1.Line (cx, cy)-(X, Y)
    Prolongs ax, ay, bx, by, cx, cy
    Prolongs X, Y, bx, by, cx, cy
    Escribeds ax, ay, bx, by, cx, cy
    Escribeds X, Y, bx, by, cx, cy
    Incircle ax, ay, bx, by, cx, cy
    Incircle X, Y, bx, by, cx, cy
    Orthocenter ax, ay, bx, by, cx, cy
    Orthocenter X, Y, bx, by, cx, cy
    Circum ax, ay, bx, by, cx, cy
    Circum X, Y, bx, by, cx, cy
    Circ qxi, qyi, qebx, qeby, qecx, qecy, 13
    Circ xi, yi, ebx, eby, ecx, ecy, 13
    qxi = xi: qyi = yi
    qebx = ebx: qeby = eby
    qecx = ecx: qecy = ecy
    ax = X: ay = Y
    Exit Sub
  End If

  d2 = Sqr((bx - X) ^ 2 + (by - Y) ^ 2)
  If d2 < 1 Then
    Form1.Line (bx, by)-(ax, ay)
    Form1.Line (cx, cy)-(bx, by)
    Form1.Line (X, Y)-(ax, ay)
    Form1.Line (cx, cy)-(X, Y)
    Prolongs ax, ay, bx, by, cx, cy
    Prolongs ax, ay, X, Y, cx, cy
    Escribeds ax, ay, bx, by, cx, cy
    Escribeds ax, ay, X, Y, cx, cy
    Incircle ax, ay, bx, by, cx, cy
    Incircle ax, ay, X, Y, cx, cy
    Orthocenter ax, ay, bx, by, cx, cy
```

```
    Orthocenter ax, ay, X, Y, cx, cy
    Circum ax, ay, bx, by, cx, cy
    Circum ax, ay, X, Y, cx, cy
    Circ qxi, qyi, qebx, qeby, qecx, qecy, 13
    Circ xi, yi, ebx, eby, ecx, ecy, 13
    qxi = xi: qyi = yi
    qebx = ebx: qeby = eby
    qecx = ecx: qecy = ecy
    bx = X: by = Y
    Exit Sub
  End If

  d3 = Sqr((cx - X) ^ 2 + (cy - Y) ^ 2)
  If d3 < 1 Then
    Form1.Line (cx, cy)-(bx, by)
    Form1.Line (ax, ay)-(cx, cy)
    Form1.Line (X, Y)-(bx, by)
    Form1.Line (ax, ay)-(X, Y)
    Prolongs ax, ay, bx, by, cx, cy
    Prolongs ax, ay, bx, by, X, Y
    Escribeds ax, ay, bx, by, cx, cy
    Escribeds ax, ay, bx, by, X, Y
    Incircle ax, ay, bx, by, cx, cy
    Incircle ax, ay, bx, by, X, Y
    Orthocenter ax, ay, bx, by, cx, cy
    Orthocenter ax, ay, bx, by, X, Y
    Circum ax, ay, bx, by, cx, cy
    Circum ax, ay, bx, by, X, Y
    Circ qxi, qyi, qebx, qeby, qecx, qecy, 13
    Circ xi, yi, ebx, eby, ecx, ecy, 13
    qxi = xi: qyi = yi
    qebx = ebx: qeby = eby
    qecx = ecx: qecy = ecy
    cx = X: cy = Y
  End If
End Sub

'-----------------------------------------------------
'      Action corresponding to the movement of mouse
'-----------------------------------------------------
Private Sub Form_MouseMove(Button As Integer, Shift As Integer, X As Single, Y As Single)
  If k = 1 Then Exit Sub

  d1 = Sqr((ax - X) ^ 2 + (ay - Y) ^ 2)
  If d1 < 1 Then
    Form1.Line (ax, ay)-(bx, by)
    Form1.Line (cx, cy)-(ax, ay)
    Form1.Line (X, Y)-(bx, by)
    Form1.Line (cx, cy)-(X, Y)
    Prolongs ax, ay, bx, by, cx, cy
    Prolongs X, Y, bx, by, cx, cy
    Escribeds ax, ay, bx, by, cx, cy
    Escribeds X, Y, bx, by, cx, cy
    Incircle ax, ay, bx, by, cx, cy
```

```
    Incircle X, Y, bx, by, cx, cy
    Orthocenter ax, ay, bx, by, cx, cy
    Orthocenter X, Y, bx, by, cx, cy
    Circum ax, ay, bx, by, cx, cy
    Circum X, Y, bx, by, cx, cy
    Circ qxi, qyi, qebx, qeby, qecx, qecy, 13
    Circ xi, yi, ebx, eby, ecx, ecy, 13
    qxi = xi: qyi = yi
    qebx = ebx: qeby = eby
    qecx = ecx: qecy = ecy
    ax = X: ay = Y
    Exit Sub
End If

d2 = Sqr((bx - X) ^ 2 + (by - Y) ^ 2)
If d2 < 1 Then
    Form1.Line (bx, by)-(ax, ay)
    Form1.Line (cx, cy)-(bx, by)
    Form1.Line (X, Y)-(ax, ay)
    Form1.Line (cx, cy)-(X, Y)
    Prolongs ax, ay, bx, by, cx, cy
    Prolongs ax, ay, X, Y, cx, cy
    Escribeds ax, ay, bx, by, cx, cy
    Escribeds ax, ay, X, Y, cx, cy
    Incircle ax, ay, bx, by, cx, cy
    Incircle ax, ay, X, Y, cx, cy
    Orthocenter ax, ay, bx, by, cx, cy
    Orthocenter ax, ay, X, Y, cx, cy
    Circum ax, ay, bx, by, cx, cy
    Circum ax, ay, X, Y, cx, cy
    Circ qxi, qyi, qebx, qeby, qecx, qecy, 13
    Circ xi, yi, ebx, eby, ecx, ecy, 13
    qxi = xi: qyi = yi
    qebx = ebx: qeby = eby
    qecx = ecx: qecy = ecy
    bx = X: by = Y
    Exit Sub
End If

d3 = Sqr((cx - X) ^ 2 + (cy - Y) ^ 2)
If d3 < 1 Then
    Form1.Line (cx, cy)-(bx, by)
    Form1.Line (ax, ay)-(cx, cy)
    Form1.Line (X, Y)-(bx, by)
    Form1.Line (ax, ay)-(X, Y)
    Prolongs ax, ay, bx, by, cx, cy
    Prolongs ax, ay, bx, by, X, Y
    Escribeds ax, ay, bx, by, cx, cy
    Escribeds ax, ay, bx, by, X, Y
    Incircle ax, ay, bx, by, cx, cy
    Incircle ax, ay, bx, by, X, Y
    Orthocenter ax, ay, bx, by, cx, cy
    Orthocenter ax, ay, bx, by, X, Y
    Circum ax, ay, bx, by, cx, cy
```

```
    Circum ax, ay, bx, by, X, Y
    Circ qxi, qyi, qebx, qeby, qecx, qecy, 13
    Circ xi, yi, ebx, eby, ecx, ecy, 13
    qxi = xi: qyi = yi
    qebx = ebx: qeby = eby
    qecx = ecx: qecy = ecy
    cx = X: cy = Y
  End If
End Sub

'-----------------------------------------
'     Extending of each side both directions
'-----------------------------------------
Private Sub Prolongs(x1, y1, x2, y2, x3, y3)
  Prolong x1, y1, x2, y2
  Prolong x2, y2, x3, y3
  Prolong x3, y3, x1, y1
End Sub

Private Sub Prolong(jx1, jy1, jx2, jy2)
  x1 = jx1: y1 = jy1

  x2 = jx2: y2 = jy2
  Form1.DrawWidth = 1
  If Abs(x2 - x1) > 0.001 Then
     m = (y2 - y1) / (x2 - x1)

    If Abs(m) <= 0.5 Then
      If x1 > x2 Then
        dmx = x1: x1 = x2: x2 = dmx
        dmy = y1: y1 = y2: y2 = dmy
      End If
      px1 = sx
      py1 = m * (sx - x1) + y1
      px2 = ex
      py2 = m * (ex - x1) + y1
      Form1.DrawStyle = 2
      Form1.Line (px1, py1)-(x1, y1), QBColor(2)
      Form1.Line (x2, y2)-(px2, py2), QBColor(2)
      Form1.DrawStyle = 0
    End If

    If Abs(m) > 0.5 Then
      If y1 > y2 Then
        dmx = x1: x1 = x2: x2 = dmx
        dmy = y1: y1 = y2: y2 = dmy
      End If
      qy1 = sy
      qx1 = x1 + (sy - y1) / m
      qy2 = ey
      qx2 = x1 + (ey - y1) / m
      Form1.DrawStyle = 2
      Form1.Line (qx1, qy1)-(x1, y1), QBColor(2)
      Form1.Line (x2, y2)-(qx2, qy2), QBColor(2)
```

```
      Form1.DrawStyle = 0
    End If
  End If
End Sub

'-----------------------------------------
'     Presenting of the excenters of a triangle
'-----------------------------------------
Private Sub Escribeds(xa, ya, xb, yb, xc, yc)
  Escribed xa, ya, xb, yb, xc, yc
  Escribed xb, yb, xc, yc, xa, ya
  ebx = xia: eby = yia
  Escribed xc, yc, xa, ya, xb, yb
  ecx = xia: ecy = yia
End Sub

Private Sub Escribed(xa, ya, xb, yb, xc, yc)
  ux = xb - xa: uy = yb - ya
  vx = xc - xa: vy = yc - ya
  wx = xc - xb: wy = yc - yb
  d = ux * vy - vx * uy
  If Abs(d) > 0.01 Then
    a = Sqr(wx ^ 2 + wy ^ 2)
    b = Sqr(vx ^ 2 + vy ^ 2)
    c = Sqr(ux ^ 2 + uy ^ 2)
    s = (a + b + c) / 2
    Ls = Sqr(s * (s - a) * (s - b) * (s - c))
    ra1 = Ls / (s - a)
    p1 = ya * xb - yb * xa
    Lf = uy * xc - ux * yc + p1
    e = -Lf / Abs(Lf)
    p2 = ya * xc - yc * xa
    Lg = vy * xb - vx * yb + p2
    f = -Lg / Abs(Lg)
    q1 = p1 + e * ra1 * c
    q2 = p2 + f * ra1 * b
    xia = (q1 * vx - q2 * ux) / (ux * vy - uy * vx)
    yia = (q1 * vy - q2 * uy) / (ux * vy - uy * vx)
    Form1.DrawWidth = 5
    Form1.PSet (xia, yia), vbRed
    Form1.DrawWidth = 1
    Form1.Circle (xia, yia), ra1, QBColor(6)
  End If
End Sub

'-----------------------------------------
'     Presenting of the incenter of a triangle
'-----------------------------------------
Private Sub Incircle(xa, ya, xb, yb, xc, yc)
  ux = xb - xa: uy = yb - ya
  vx = xc - xa: vy = yc - ya
  wx = xc - xb: wy = yc - yb
  d = ux * vy - vx * uy
  If Abs(d) > 0.01 Then
```

```
      a = Sqr(wx ^ 2 + wy ^ 2)
      b = Sqr(vx ^ 2 + vy ^ 2)
      c = Sqr(ux ^ 2 + uy ^ 2)
      s = (a + b + c) / 2
      Ls = Sqr(s * (s - a) * (s - b) * (s - c))
      r = Ls / s
      p1 = ya * xb - yb * xa
      Lf = uy * xc - ux * yc + p1
      e = -Lf / Abs(Lf)
      p2 = ya * xc - yc * xa
      Lg = vy * xb - vx * yb + p2
      f = -Lg / Abs(Lg)
      q1 = p1 + e * r * c
      q2 = p2 + f * r * b
      xi = (q1 * vx - q2 * ux) / (ux * vy - uy * vx)
      yi = (q1 * vy - q2 * uy) / (ux * vy - uy * vx)
      Form1.DrawWidth = 5
      Form1.PSet (xi, yi), vbGreen
      Form1.DrawWidth = 1
      Form1.Circle (xi, yi), r, QBColor(2)
   End If
End Sub

'---------------------------------------------
'     Presenting of the orthocenter of a triangle
'---------------------------------------------
Private Sub Orthocenter(jxa, jya, jxb, jyb, jxc, jyc)
   xa = jxa: ya = jya
   xb = jxb: yb = jyb
   xc = jxc: yc = jyc
   ux = xb - xa: uy = yb - ya
   vx = xa - xc: vy = ya - yc
   wx = xc - xb: wy = yc - yb
   a = Sqr(wx ^ 2 + wy ^ 2)
   b = Sqr(vx ^ 2 + vy ^ 2)
   c = Sqr(ux ^ 2 + uy ^ 2)
   ah = b ^ 2 + c ^ 2 - a ^ 2
   bh = c ^ 2 + a ^ 2 - b ^ 2
   ch = a ^ 2 + b ^ 2 - c ^ 2
   If ah >= 0 And bh >= 0 And ch >= 0 Then
      p1 = xb * yc - xc * yb: q1 = -wy * ya - wx * xa
      p2 = xc * ya - xa * yc: q2 = -vy * yb - vx * xb
      p3 = xa * yb - xb * ya: q3 = -uy * yc - ux * xc
      xd = (p1 * wy - q1 * wx) / a ^ 2
      yd = (-p1 * wx - q1 * wy) / a ^ 2
      xe = (p2 * vy - q2 * vx) / b ^ 2
      ye = (-p2 * vx - q2 * vy) / b ^ 2
      xf = (p3 * uy - q3 * ux) / c ^ 2
      yf = (-p3 * ux - q3 * uy) / c ^ 2
      xh = (q2 * wy - q1 * vy) / (wx * vy - wy * vx)
      yh = (q1 * vx - q2 * wx) / (wx * vy - wy * vx)
      Form1.DrawStyle = 2
      Form1.Line (xa, ya)-(xd, yd)
      Form1.Line (xb, yb)-(xe, ye)
```

```
    Form1.Line (xc, yc)-(xf, yf)
    Form1.DrawStyle = 0
    Form1.DrawWidth = 5
    Form1.PSet (xh, yh), vbRed
    Form1.DrawWidth = 1
  Else
    If bh < 0 Then
      dm = xa: xa = xb: xb = dm
      dm = ya: ya = yb: yb = dm
    End If

    If ch < 0 Then
      dm = xa: xa = xc: xc = dm
      dm = ya: ya = yc: yc = dm
    End If

    ux = xb - xa: uy = yb - ya
    vx = xa - xc: vy = ya - yc
    wx = xc - xb: wy = yc - yb
    d = wx * vy - wy * vx
    If Abs(d) > 0.001 Then
      a = Sqr(wx ^ 2 + wy ^ 2)
      b = Sqr(vx ^ 2 + vy ^ 2)
      c = Sqr(ux ^ 2 + uy ^ 2)
      p1 = xb * yc - xc * yb: q1 = -wy * ya - wx * xa
      q2 = -vy * yb - vx * xb
      xd = (p1 * wy - q1 * wx) / a ^ 2
      yd = (-p1 * wx - q1 * wy) / a ^ 2
      xh = (q2 * wy - q1 * vy) / d
      yh = (q1 * vx - q2 * wx) / d
      H_Prolong xb, yb, xa, ya
      H_Prolong xc, yc, xa, ya
      Form1.DrawStyle = 2
      Form1.Line (xh, yh)-(xd, yd), vbBlue
      Form1.Line (xb, yb)-(xh, yh), vbBlue
      Form1.Line (xc, yc)-(xh, yh), vbBlue
      Form1.DrawStyle = 0
      Form1.DrawWidth = 5
      Form1.PSet (xh, yh), vbRed
      Form1.DrawWidth = 1
    End If
  End If
End Sub

'----------------------------------------------------------------
'   Extending of two sides adjacent at the vertex with an obtuse angle
'----------------------------------------------------------------
Private Sub H_Prolong(x1, y1, x2, y2)
  Form1.DrawWidth = 1
  l = Sqr((x1 - x2) ^ 2 + (y1 - y2) ^ 2)

  ra(1) = Sqr((x1 - sx) ^ 2 + (y1 - sy) ^ 2)
  ra(2) = Sqr((x1 - ex) ^ 2 + (y1 - sy) ^ 2)
  ra(3) = Sqr((x1 - ex) ^ 2 + (y1 - ey) ^ 2)
```

```
    ra(4) = Sqr((x1 - sx) ^ 2 + (y1 - ey) ^ 2)

    r = ra(1)
    For i = 2 To 4
      If ra(i) > r Then r = ra(i)
    Next i


    p = r / l
    tx = (1 - p) * x1 + p * x2
    ty = (1 - p) * y1 + p * y2
    Form1.DrawStyle = 2
    Form1.Line (x2, y2)-(tx, ty), QBColor(2)
    Form1.DrawStyle = 0
End Sub

'---------------------------------------------
'     Presenting of the circumcenter of a triangle
'---------------------------------------------
Private Sub Circum(xa, ya, xb, yb, xc, yc)
    mx = (xa + xb) / 2: my = (ya + yb) / 2
    nx = (xa + xc) / 2: ny = (ya + yc) / 2
    ux = xb - xa: uy = yb - ya
    vx = xc - xa: vy = yc - ya
    wx = xc - xb: wy = yc - yb
    d = ux * vy - vx * uy
    If Abs(d) > 0.01 Then
      a = Sqr(wx ^ 2 + wy ^ 2)
      b = Sqr(vx ^ 2 + vy ^ 2)
      c = Sqr(ux ^ 2 + uy ^ 2)
      s = (a + b + c) / 2
      Ls = Sqr(s * (s - a) * (s - b) * (s - c))
      r = (a * b * c) / (4 * Ls)
      p = ux * mx + uy * my
      q = vx * nx + vy * ny
      xo = (p * vy - q * uy) / d
      yo = (q * ux - p * vx) / d
      Form1.DrawWidth = 5
      Form1.PSet (xo, yo), vbBlue
      Form1.DrawWidth = 1
      Form1.Circle (xo, yo), r, vbBlue
    End If
End Sub

'---------------------------------------------
'     Drawing the circle determined by three points
'---------------------------------------------
Private Sub Circ(pxa, pya, pxb, pyb, pxc, pyc, col)
    za = pxa ^ 2 + pya ^ 2
    zb = pxb ^ 2 + pyb ^ 2
    zc = pxc ^ 2 + pyc ^ 2
    s = pxa * pyb + pxb * pyc + pxc * pya - pxa * pyc - pxb * pya - pxc * pyb
    If Abs(s) > 0.001 Then
      t = -za * pyb - zb * pyc - zc * pya + za * pyc + zb * pya + zc * pyb
```

```
    u = za * pxb + zb * pxc + zc * pxa - za * pxc - zb * pxa - zc * pxb
    v = -za * pxb * pyc - zb * pxc * pya - zc * pxa * pyb + za * pxc * pyb
          + zb * pxa * pyc + zc * pxb * pya
    r = Sqr(t ^ 2 / (4 * s ^ 2) + u ^ 2 / (4 * s ^ 2) - v / s)
    If r < 1000 Then
      Form1.DrawWidth = 5
      Form1.PSet (-t / (2 * s), -u / (2 * s)), QBColor(col)
      Form1.DrawWidth = 1
      Form1.Circle (-t / (2 * s), -u / (2 * s)), r, QBColor(col)
    End If
  End If
End Sub
```