Mem. Fac. Educ., Kagawa Univ. II, **57** (2007), 17-27

On the Beurling Convolution Algebra III

K. ANZAI, K. HORIE and S. KOIZUMI

Kagawa University, Takamatsu National College of Technology and Keio University

Abstract

We treat the Beurling convolution algebra. Especially, Theorem Ⅳ and Theorem Ⅷ in [4] are extended to the case of n-dimensional euclidean space R^n.

1. Introduction

A. Beurling [**4**] considered the theorems concerning a class of functions of R^1, each member of which is the Fourier transform of an integrable function. The purpose of this paper is to extend his results to the class of functions on R^n.

Let us start to set notations, definitions and theorems, which we shall ask for, according to A. Beurling [**4**]. Let G be a locally compact Abelian group with the invariant measure dx. We consider a normed family Ω of strictly positive functions $\omega(x)$ on G which are measurable with respect to dx, and furthermore, together with the norm $N(\omega)$, satisfy the following conditions

(Ⅰ) For each $\omega \in \Omega$, $N(\omega)$, takes a finite value,

$$0 < \int \omega \, dx \leq N(\omega).$$

(Ⅱ) If λ is a positive number and $\omega \in \Omega$, then $\lambda \omega \in \Omega$ and

$$N(\lambda \omega) = \lambda N(\omega).$$

(Ⅲ) If $\omega_1, \omega_2 \in \Omega$, then sum $\omega_1 + \omega_2$ as well as the convolution $\omega_1 * \omega_2$ are also in Ω and

$$N(\omega_1+\omega_2)\leq N(\omega_1)+N(\omega_2).$$

$$N(\omega_1 * \omega_2)\leq N(\omega_1)N(\omega_2).$$

(IV) Ω is complete under the norm N in the sense that for any sequence $\{\omega_n\}_1^\infty \subset \Omega$ such that $\sum_1^\infty N(\omega_n)<\infty$, it is satisfied that $\omega=\sum_1^\infty \omega_n$ is in Ω and

$$N(\omega)\leq \sum_1^\infty N(\omega_n).$$

We associate with each , the Banach space $L^2_{\omega^{-1}}(G)$ of measurable functions on G and having the norms

$$\| F \|_{L^2_{\omega^{-1}}} = \left(\int_G \omega\, dx \int_G \frac{|F|^2}{\omega}\, dx \right)^{1/2}$$

From these spaces, we obtain set of functions $A^2=A^2(G, \Omega)$ by setting

$$A^2(G) = A^2(G, \Omega) = \bigcup_{\omega\in\Omega} L^2_{\omega^{-1}}(G)$$

and define

$$\| F \| = \| F \|_{A^2} = \inf_{\omega\in\Omega} \| F \|_{L^2_{\omega^{-1}}}.$$

Now A. Beurling [4] proved the following Theorem I, II, III, IV, and V.

Theorem I. In the norm (1.3), $A^2(G)$ is the Banach algebra under addition and convolution, and we have

(1.1) $$\| F_1 * F_2 \|_{A^2} \leq \| F_1 \|_{A^2} \| F_2 \|_{A^2}$$

where F_1 and F_2 belong to $A^2(G)$.

A.Beurling [4] considered the theorems concerning a class of functions of R^1, each member of which is the Fourier transform of an integrable function.

Let us remark that $A^2(R^1)\subset L^1(R^1)$ by the inequality

$$\int_{R^1} |F|\, dx \leq \left(\int_{R^1} \omega dx \int_{R^1} \frac{|F|^2}{\omega} dx \right)^{\frac{1}{2}}$$

and so the Fourier transform of $F \in A^2(R^1)$ is well defined

Let us define the normed ring

$$\tilde{A}^2(R^1) = \{f : f = \hat{F}, F \in A^2\}$$

with norm

$$\|f\|_{\tilde{A}^2} = \|F\|_{A^2}.$$

Let us also define the notation as follows

$$\eta(\alpha, f) = \sqrt{\frac{1}{2\pi}\int_{R^1} |f(t+\alpha) - f(t)|^2 dt}\,.$$

Let us introduce the following integral

$$A(f) = \int_0^\infty \eta(\alpha, f)\frac{d\alpha}{|\alpha|^{3/2}}.$$

This integral $A(f)$ will be an important tool for the analysis of $\tilde{A}^2$.

Theorem II. A function f belongs to the normed ring $\tilde{A}^2(R^1, \Omega)$ if and only if the following conditions are satisfied

(a) f is continuous,

(b) $\lim_{|t|\to\infty} f(t) = 0$,

(c) $A(f) < \infty$.

Under these conditions, f is the Fourier transform of an $F \in A^2(R^1, \Omega)$, and the following inequalities hold

$$\|f\|_{\tilde{A}^2} \leq A(f) \leq 5\,\|F\|_{A^2},$$

provided $f \neq 0$.

Theorem III. Let g be a continuous function and a contraction of the series $\sum_{\nu=1}^{N} f_\nu$ where each f_ν belongs to $\tilde{A}^2 = \tilde{A}^2(R^1, \Omega)$. Then we have

$$g \in \tilde{A}^2 \text{ and } \|g\|_{\tilde{A}^2} \leq k\sum_{\nu=1}^{N} \|f_\nu\|_{\tilde{A}^2}$$

where $k(<5)$ is a constant depending only on the space.

If, in a sequence of continuous functions $\{g_m\}$, each function is a contraction of series $\sum_{\nu=1}^{N} f_\nu$, then the assumption

$$\lim_{m\to\infty} M(g_m) = 0$$

implies

$$\lim_{m\to\infty} \|g_m\|_{\bar{A}^2} = 0.$$

Where $M(g)$ denotes the supremum norm of g.

Next let us consider the family Ω_1 defined as the subset of Ω consisting of functions with the property

$$\omega(0) = \lim_{x\to 0} \omega(x) < \infty.$$

The norm in Ω_1 will be defined as

$$N(\omega) = \omega(0) + \int_{R^n} \omega dx.$$

We can define the Banach algebra $\mathscr{A}_1^2 = \mathscr{A}_1^2(R^n, \Omega_1)$.

We shall denote by $B^2 = B^2(R^n)$ and by $B_1^2 = B_1^2(R^n)$ the spaces of measurable functions on R^n which are L^2 over compact sets and have the norms

$$\|\Phi\|_{B^2} = \sup_{r>0} \left\{ \frac{\int_{|x|\le r} |\Phi|^2 dx}{V_n(r)} \right\}^{\frac{1}{2}},$$

$$\|\Phi\|_{B_1^2} = \sup_{r>0} \left\{ \frac{\int_{|x|\le r} |\Phi|^2 dx}{1 + V_n(r)} \right\}^{\frac{1}{2}},$$

where $V_n(r)$ denotes the volume of the sphere $|x| \le r$ in R^n.

Theorem IV. $B^2 = B^2(R^n)$ is the conjugate space of the algebra $A^2 = A^2(R^n, \Omega)$. Similarly, $B_1^2 = B_1^2(R^n)$ is the conjugate space of the algebra $\mathscr{A}_1^2 = \mathscr{A}_1^2(R^n, \Omega_1)$.

Theorem V. The space $\mathscr{A}_1^2 = \mathscr{A}_1^2(R^n, \Omega_1)$ is the intersection of $A^2 = A^2(R^1, \Omega_1)$ and $L^2(R^1)$, and the norms in these spaces satisfy the inequalities

(1.2) $\quad \|F\|_{\mathscr{A}_1^2} > \|F\|_{A^2}$

(1.3) $\quad \|F\|_{\mathscr{A}_1^2} > \|F\|_{L^2}$

(1.4) $\quad \|F\|_{\mathscr{A}_1^2} < \|F\|_{A^2} + \|F\|_{L^2}$

2. The class of functions on R^n

The purpose of this paper is to extend results of Beurling to the class of functions on n-dimensional euclidean space R^n.

Let us remark that $A^2(R^n) \subset L^1(R^n)$ by the inequality

$$\int_{R^n} |F|\,dx \leq \left(\int_{R^n} \omega dx \int_{R^n} \frac{|F|^2}{\omega} dx\right)^{\frac{1}{2}}$$

and so the Fourier transform of $F \in A^2(R^n)$ is well defined.

Let us define the normed ring

$$\tilde{A}^2 = \left\{ f : f = \hat{F},\ F \in A^2 \right\}$$

with norm

$$\|f\|_{\tilde{A}^2} = \|f\|_{A^2}.$$

Let us also define the notation as follows

$$\eta(\alpha, f) = \sqrt{\left(\frac{1}{2\pi}\right)^n \int_{R^n} |\Delta^n_\alpha f(t)|^2 dt}$$

where $\Delta^n_\alpha f$ is the vector difference along α, that is

$$\Delta^n_\alpha f(t) = \sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!} f\ (t + (n-k)\,\alpha).$$

and

$$A(f) = \int_{R^n} \eta(\alpha, f) \frac{d\alpha}{|\alpha|^{3n/2}}.$$

In the preceding paper [**1**], we proved the following theorem.

Theorem 1. A function f belongs to the normed ring $\tilde{A}^2(R^n)$ if and only if the following conditions are satisfied

(a) f is continuous,

(c) $\lim_{|t| \to \infty} f(t) = 0$,

(c) $A(f) < \infty$.

Under these conditions, f is the Fourier transform of $F \in A^2(R^n)$, and the following inequalities hold

$$c_n \| F \|_{A^2} \leq A(f) \leq d_n \| F \|_{A^2},$$

provided $f \neq 0$, where c_n and d_n are positive constants which are independent of f.

It is convenient to give at this instance some accounts of their basic properties, for the sake of completeness and the reasons of indispensable for further studies. These can be done by running the same lines as A.Beurling [**4**] .

(1) The normed ring $\tilde{A}^2(R^n)$ is complete and separable.

Since A.Beurling [**4**] proved that in Theorem I that the $\tilde{A}^2(R^n)$ is the Banach algebra under addition and convolution. According to these results, the $\tilde{A}^2(R^n)$ is complete normed ring under pointwise addition and multiplication. Furthermore it is clear that $\tilde{A}^2(R^n)$ is separable and so $\tilde{A}^2(R^n)$ does too.

(2) The normed ring $\tilde{A}^2(R^n)$ satisfies the principle uniform contraction with certain constant.

We shall generalize Theorem Ⅲ to functions on n-dimensional euclidean space R^n and this is the theorem in the preceding paper [**1**] without proof.

Theorem 2. Let g be a continuous function and a contraction of the series $\sum_{\nu=1}^{N} f_\nu$, where each f_ν belongs to $\tilde{A}^2(R^n)$. Then we have

$$g \in \tilde{A}^2(R^n) \text{ and } \| g \|_{\tilde{A}^2} \leq \frac{d_n}{c_n} \sum_{\nu=1}^{N} f_\nu ,$$

where constants c_n and d_n are those of Theorem 1.

If, in a sequence of continuous functions $\{g_m\}$, each function is a contraction of series $\sum_{\nu=1}^{N} f_\nu$, then the assumption

$$\lim_{m \to \infty} M(g_m) = 0$$

implies

$$\lim_{m \to \infty} \| g_m \|_{\tilde{A}^2} = 0$$

This theorem is the essential tool for further applications and so we shall give its proof.

Proof. Let us suppose that g be a contraction of the series $\sum_{\nu=1}^{N} f_\nu$, $f_\nu \in \tilde{A}^2(R^n)$. That is, for any t and α, the following properties

(2.1) $$|g(t)| \leq \sum_{\nu=1}^{N} |f_\nu(t)|$$

and

(2.2) $$|\Delta_\alpha^n g(t)| \leq \sum_{\nu=1}^{N} |\Delta_\alpha^n f_\nu(t)|$$

satisfied.

Then we have by the inequality (2.1),

$$\lim_{|t|\to\infty} g(t) = 0$$

and by the inequality (2.2) ,

$$\eta(\alpha, g) \leq \sum_{\nu=1}^{N} \eta(\alpha, f_\nu)$$

and so

$$A(g) \leq \sum_{\nu=1}^{N} A(f_\nu).$$

Since we suppose that $g(t)$ is continuous, we can conclude that $g \in \tilde{A}^2(R^n)$ by the Theorem1 and we have

$$\|g\|_{\tilde{A}_2} \leq \frac{1}{c_n} A(g) \leq \frac{1}{c_n} \sum_{\nu=1}^{N} A(f_\nu) \leq \frac{d_n}{c_n} \sum_{\nu=1}^{N} \|f_\nu\|_{\tilde{A}_2}.$$

Next let us suppose that $\lim_{m\to\infty} M(g_m) = 0$, then we have

$$|\Delta_\alpha^n g_m(t)| \leq \sum_{\nu=1}^{N} |\Delta_\alpha^n f_\nu(t)| \in L^2(R^n)$$

while

$$\lim_{m\to\infty} \Delta_\alpha^n g_m(t) = 0$$

for fixed α and t. Hence by the Lebesgue theorem of dominated convergence,

$$\lim_{m\to\infty} \eta(\alpha, g_m) = 0.$$

Similarly, we have

$$\frac{\eta(\alpha,g_m)}{|\alpha|^{3n/2}} \leq \sum_{\nu=1}^{N} \frac{\eta(\alpha,f_\nu)}{|\alpha|^{3n/2}} \in L^2(R^n)$$

and we may apply the same theorem to the integral $A(g_m)$ obtaining

$\lim_{m\to\infty} A(g_m)=0$ and then by Theorem 1 we have $\lim_{m\to\infty} \|g_m\|_{\tilde{A}^2}=0.$

Thus we proved the theorem.

(3) The $\tilde{A}^2(R^n)$ satisfy the following property

(2.3) $$\sup_{\|g\|_{\tilde{A}^2}\leq 1} \frac{|g(t)|}{\|g\|_{\tilde{A}^2}}=1, \qquad (\forall t\in R^n).$$

According to A.Beurling [**4**] , this proved as follows. Let us set $G(x)=e^{it_0x}\omega_0(x)$ for any $t_0\in R^n$ with normalized $\omega_0\in\Omega$ and its Fourier transform

$$g(t)=\int e^{-itx}G(x)\,dx.$$

Then we have

$$g(t_0)=\int\omega_0(x)\,dx=1,$$

and

$$\|g\|_{\tilde{A}^2}^2=\|G\|_{A^2}^2=\inf_{\omega\in\Omega}\int\omega(x)\,dx\int\frac{\|G\|^2}{\omega}dx$$

$$\leq\int\omega_0(x)\,dx\int\frac{|G|^2}{\omega_0}dx=\left(\int\omega_0 dx\right)^2=1.$$

On the other hand, we have

$$|g(t_0)|\leq M(g)\leq\int|G|\,dx\leq\|G\|_{A_2}\|g\|_{\tilde{A}_2}.$$

Therefore we have $|g(t_0)|=\|g\|_{\tilde{A}_2}=1$ which shows the property (2.3) at $t=t_0$.

We observe that at once (2.3) imply

(2.4) $$M(g)\leq\|g\|_{\tilde{A}_2}.$$

But this proved immediately by the definition of $\tilde{A}^2(R^n)$.

We shall generalize Theorem V to functions on n-dimensional euclidean space R^n and this is the theorem in the preceding paper [**1**] without proof.

Theorem 3. The space $\mathscr{A}_1^2 = \mathscr{A}_1^2(R^n, \Omega_1)$ is the intersection of $A^2 = A^2(R^n, \Omega)$ and $L^2(R^n)$, and the norms in these spaces satisfy the inequalities

(2.5) $\quad \|F\|_{\mathscr{A}_1^2} > \|F\|_{A^2}$

(2.6) $\quad \|F\|_{\mathscr{A}_1^2} > \|F\|_{L^2}$

(2.7) $\quad \|F\|_{\mathscr{A}_1^2} < \|F\|_{A^2} + \|F\|_{L^2}$

Proof. The relation (2.5) is a consequence of the fact that Ω_1 is subset of Ω. As to the proof of (2.6) we recall that

$$\|F\|_{A^2} = \inf_{\omega\in\Omega}\left(\left(\int_{R^n}\omega(|x|)dx\right)\int_{R^n}\frac{|F(x)|^2}{\omega(|x|)}dx\right)^{\frac{1}{2}}$$

$$\leq \inf_{\omega\in\Omega_1}\left(\left(\omega(0)+\int_{R^n}\omega(|x|)dx\right)\int_{R^n}\frac{|F(x)|^2}{\omega(|x|)}dx\right)^{\frac{1}{2}}$$

By Theorem Ⅳ, for $\Phi \in B_1^2(R^n) = \left(\mathscr{A}_1^2\ (R^n)\right)^*$,

$$\|\Phi\|_{B_1^2} = \sup_{r>0}\left(\frac{\int_{|x|\leq r}|\Phi(x)|^2dx}{1+V_n(r)}\right) \leq \left(\int_{R^n}|\Phi(x)|^2dx\right)^{\frac{1}{2}} = \|\Phi\|_{L_2}$$

Hence, on taking $\Phi = \bar{F}$,

$$\|F\|_{L_2}^2 = \int_{R^n}F\Phi\,dx \leq \|F\|_{\mathscr{A}_1^2}\|\Phi\|_{B_1^2} < \|F\|_{\mathscr{A}_1^2}\|\Phi\|_{L^2},$$

and (2.6) follows. For $F\in L^2(R^n)\cap A^2(R^n)$ We set

$$a = \|F\|_{L^2},\ b = \left(\int_{R^n}\frac{|F|^2}{\omega}dx\right)^{\frac{1}{2}},$$

where $\omega\in\Omega$ is normalized and b is a number close to $\|F\|_{A^2}$.
Clearly

$$\omega_1(|x|) = \frac{ab\,\omega(|x|)}{a+b\,\omega(|x|)}$$

belongs to Ω_1 since $\omega_1(|x|)$ is nonincreasing function of $|x|$.
We have

$$N(\omega_1) = \omega_1(0) + \int_{R^n}\omega_1(|x|)dx$$

$$\leq \frac{ab\,\omega(0)}{a+b\,\omega(0)} + \int_{R^n}\frac{ab\,\omega(|x|)}{a+b\,\omega(|x|)}dx < a+b$$

Thus we obtain that

$$\|F\|_{\mathscr{A}_1^2} \leq \left(\omega(0)+\int_{R^n}\omega_1(|x|)dx\right)\int_{R^n}\frac{|F(x)|^2}{\omega_1(|x|)}dx$$

$$< (a+b)\left(\frac{1}{b}\int_{R^n}\frac{|F(x)|^2}{\omega(|x|)}dx+\frac{1}{a}\int_{R^n}|F(x)|^2dx\right)=(a+b)^2$$

which proves

$$\|F\|_{\mathscr{A}_1^2} < \|F\|_{A^2}+\|F\|_{L^2}$$

That the strict inequality actually holds for $F \neq 0$ is easily established.
By inequality (2.5) and (2.6) ,

$$\mathscr{A}_1^2(R^n) \subset A^2(R^n) \cap L^2(R^n).$$

By inequality (2.7),

$$\mathscr{A}_1^2(R^n) \supset A^2(R^n) \cap L^2(R^n).$$

Thus Theorem 3 is proved.

References

[1] K.Anzai, K.Horie and S. Koizumi, On the A. Beurling convolution algebra, Tokyo J. Math. **19** (1996), 85-98.
[2] K.Anzai, K.Horie and S. Koizumi, On the Beurling convolution algebra II, Tokyo J. Math. **26** (2003), 275-300.
[3] A.Beurling, On the spectral synthesis of bounded functions, Acta Math. **81** (1949), 225-238.
[4] A.Beurling, Construction and analysis of some convolution algebras, Ann. Inst. Fourier, **14** (1964), 1-32.
[5] A.Beurling, Collected Works of Arne Beurling, Vol.**2**, Birkhäuser (1989).

Present Addresses

Kazuo ANZAI
Department of Mathematics, Kagawa University,
Saiwai-cho, Takamatsu, 760-8522, Japan

Kenji HORIE
Department of Mathematics, Takamatsu National College of Technology, Chokushi-cho, Takamatsu, 761-8058, Japan

Sumiyuki KOIZUMI
Department of Mathematics, Faculty of Sciences and Technology, Keio University, Hiyoshi, Kohoku-ku, Yokohama, 223-8522, Japan